SUPPLEMENT

TO

THE ELEMENTS

OF

EUCLID.

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PREFACE.

THE propositions contained in the following compilation are either obvious deductions from those of Euclid, or such as exhibit some remarkable properties of lines, angles, or figures, which are not to be found in Euclid's work; or, lastly, they are the geometrical solutions of many well-known problems in the different branches of Natural Philosophy. But although the propositions, which have here been collected for the use of the academical student, are of these three kinds, it has not been thought advisable to class them according to that threefold division. Designed as a supplement to the Elements of Euclid, they have been disposed according to Euclid's arrangement. And not only have those which constitute the first book been made to depend upon the first book of the Elements, and so on; but the propositions in each separate book will also be found to follow the order of the propositions of the corresponding book of Euclid. There is no necessity, therefore, for the student to wait until he has gone through Euclid's Elements, before he enters upon the perusal of this Supplement. It will, perhaps, be more to his advantage to read the original work and this, which is principally intended to supply its deficiencies, together; especially if he has the assistance of a tutor, who will point out to him those propositions which may be considered as best deserving his attention. Some regard has, indeed, been paid to the probability of such a plan being thought worthy of adoption, in the distribution of the matter of this present publication. An endeavour has been made to offer something to the notice of the reader, after almost every one of the most important propositions, in each of the books of Euclid's Elements: so that, supposing him not to advance beyond the first book, or beyond the first four books, of Euclid, a field, more or less contracted, is still open to his research, for the exploring of which he will find himself already sufficiently furnished with previous knowledge. With this view, especially, many of the following theorems and problems, which might undoubtedly have been demonstrated more concisely, if they had been put after Euclid's fifth book, have had a place assigned to them nearer to the beginning. For thus is the learner shewn how extensive an application may be made of some of the easiest propositions of Geometry; and thus is a scope afforded to the study of those, who cannot at first encounter, without reluctance, the somewhat abstruse reasonings, upon which the ancients, with so much acuteness and solidity of judgment, have founded the doctrine of proportionality.

In order to facilitate the execution of the plan here recommended, an index has been constructed, by means of which the Geometry of this Supplement may be incorporated, as it were, with that of Euclid, and the reading of both the treatises may be made to go on together.

In the demonstrations of the propositions recourse has been had to symbols. But these symbols are merely the representatives of certain words and phrases, which may be substituted for them at pleasure, so as to render the language employed strictly conformable to that of ancient Geometry. The consequent diminution of the bulk of the whole book is the least advantage which results from this use of symbols. For the demonstrations themselves are sooner read and more easily comprehended by means of these useful abbreviations, which will, in a short time, become familiar to the reader, if he is not already perfectly well acquainted with them.

It appeared to be unnecessary to print the formal and logical conclusion which belongs to every geometrical demonstration, and which consists in repeating the enuntiation of the proposition which was to be proved, and in asserting that it has been proved. This last step, is, therefore, left for the reader in all cases mentally to supply. And if some omissions of a weightier kind, and some errors, be discoverable in the following pages, it is hoped that they will be found neither too great, nor too many to be forgiven, if the general plan of the work meet with the approbation of those who are competent to decide upon it.

Trinity College,
April 27th, 1819.

AN INDEX,

Shewing the Order in which the Propositions of the following Supplement may be read along with the Propositions contained in Euclid's Elements.

BOOK I.

After	E. 9.	1.	may be read	S. 1.	1.
——	E. 10.	1.	——	S. 2.	—
——	E. 11.	1.	——	S. 3.	—
				S. 4.	—
				S. 5.	—
——	E. 16.	1.	——	S. 6.	—
——	E. 17.	1.	——	S. 7.	—
				S. 8.	—
——	E. 20.	1.	——	S. 9.	—
				S. 10.	—
				S. 11.	—
				S. 12.	—
				S. 13.	—
——	E. 21.	1.	——	S. 14.	—
——	E. 23.	1.	——	S. 15.	—
——	E. 26.	1.	——	S. 16.	—
				S. 17.	—
——	E. 27.	1.	——	S. 18.	—
——	E. 29.	1.	——	S. 19.	—
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After E. 31. 1. may be read	S. 22.	1.
	S. 23.	—
	S. 24.	—
	S. 25.	—
—— E. 32. 1. ——	S. 26.	—
	S. 27.	—
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	S. 38.	—
	S. 39.	—
—— E. 33. 1. ——	S. 40.	—
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	S. 50.	—
	S. 51.	—
	S. 52.	—
	S. 53.	—
—— E. 36. 1. ——	S. 54.	—
	S. 55.	—
	S. 56.	—

After	E. 37.	1.	may be read	S. 57.	1.
——	E. 38.	1.	——	S. 58.	—
				S. 59.	—
				S. 60.	—
				S. 61.	—
				S. 62.	—
				S. 63.	—
				S. 64.	—
——	E. 40.	1.	——	S. 65.	—
——	E. 41.	1.	——	S. 66.	—
				S. 67.	—
				S. 68.	—
				S. 69.	—
——	E. 43.	1.	——	S. 70.	—
——	E. 45.	1.	——	S. 71.	—
				S. 72.	—
——	E. 47.	1.	——	S. 73.	—
				S. 74.	—
				S. 75.	—
				S. 76.	—
				S. 77.	—

BOOK II.

After	E. 1.	2.	may be read	S. 1.	2.
——	E. 5.	2.	——	S. 2.	—
				S. 3.	—
——	E. 6.	2.	——	S. 4.	—
——	E. 7.	2.	——	S. 5.	—
				S. 6.	—
——	E. 8.	2.	——	S. 7.	—
				S. 8.	—

After	E.	10.	2.	may be read	S.	9.	2.
					S.	10.	—
					S.	11.	—
———	E.	12.	2.	———	S.	12.	—
———	E.	13.	2.	———	S.	13.	—
					S.	14.	—

BOOK III.

After	E.	1.	3.	may be read	S.	1.	3.
———	E.	3.	3.	———	S.	2.	—
———	E.	14.	3.	———	S.	3.	—
					S.	4.	—
					S.	5.	—
———	E.	16.	3.	———	S.	6.	—
					S.	7.	—
					S.	8.	—
					S.	9.	—
					S.	10.	—
					S.	11.	—
					S.	12.	—
					S.	13.	—
					S.	14.	—
					S.	15.	—
					S.	16.	—
					S.	17.	—
———	E.	17.	3.	———	S.	18.	—
———	E.	18.	3.	———	S.	19.	—
					S.	20.	—
					S.	21.	—
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After	E. 20.	3.	may be read	S. 25.	3.
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——	E. 22.	3.	——	S. 27.	—
				S. 28.	—
				S. 29.	—
				S. 30.	—
				S. 31.	—
				S. 32.	—
——	E. 23.	3.	——	S. 33.	—
——	E. 24.	3.	——	S. 34.	—
——	E. 26.	3.	——	S. 35.	—
				S. 36.	—
				S. 37.	—
				S. 38.	—
——	E. 27.	3.	——	S. 39.	—
				S. 40.	—
				S. 41.	—
——	E. 28.	3.	——	S. 42.	—
				S. 43.	—
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				S. 45.	—
——	E. 29.	3.	——	S. 46.	—
——	E. 30.	3.	——	S. 47.	—
				S. 48.	—
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——	E. 31.	3.	——	S. 50.	—
				S. 51.	—
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				S. 57.	—
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After	E. 32.	3.	may be read	S. 59.	3.
				S. 60.	—
——	E. 33.	3.	——	S. 61.	—
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				S. 65.	—
				S. 66.	—
——	E. 35.	3.	——	S. 67.	—
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				S. 87.	—
——	E. 36.	3.	——	S. 72.	—
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				S. 90.	—

After E. 36. 3. may be read	S. 91. 3. S. 92. — S. 94. —
—— E. 37. 3. ——	S. 93. — S. 95. — S. 96. — S. 97. — S. 98. — S. 99. — S. 100. —

BOOK IV.

After E. 3. 4. may be read	S. 1. 4. S. 2. —
—— E. 4. 4. ——	S. 3. — S. 4. — S. 5. — S. 6. — S. 7. — S. 8. —
—— E. 6. 4. ——	S. 9. — S. 10. — S. 11. —
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—— E. 10. 4. ——	S. 14. —
—— E. 11. 4. ——	S. 15. — S. 16. —
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					S.	3.	—
					S.	4.	—
——	E.	12.	6.	——	S.	5.	—
——	E.	13.	5.	——	S.	6.	—
——	E.	16.	5.	——	S.	7.	—
					S.	8.	—
——	E.	17.	5.	——	S.	9.	—
——	E.	18.	5.	——	S.	10.	—
					S.	11.	—
					S.	12.	—
					S.	13.	—
					S.	14.	—
					S.	16.	—
——	E.	19.	5.	——	S.	15.	—
——	E.	22.	5.	——	S.	16.	—
——	E.	23.	5.	——	S.	17.	—
——	E.	25.	5.	——	S.	19.	—
					S.	20.	—
					S.	21.	—

BOOK VI.

After	E.	1.	6.	may be read	S.	1.	6.
——	E.	3.	6.	——	S.	2.	—
——	E.	4.	6.	——	S.	3.	—
					S.	4.	—
					S.	5.	—
					S.	6.	—
					S.	7.	—

After E. 4. 6. may be read	S. 46.	6.
	S. 8.	—
	S. 9.	—
	S. 10.	—
	S. 11.	—
	S. 12.	—
	S. 13.	—
	S. 14.	—
	S. 15.	—
	S. 16.	—
	S. 20.	—
—— E. 6. 6. ——	S. 17.	—
—— E. 8. 6. ——	S. 18.	—
	S. 19.	—
—— E. 10. 6. ——	S. 21.	—
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	S. 31.	—
	S. 32.	—
	S. 44.	—
—— E. 11. 6. ——	S. 33.	—
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	S. 35.	—
	S. 36.	—
	S. 37.	—
	S. 38.	—
	S. 39.	—

After E. 12. 6. may be read	S. 40.	6.
	S. 41.	—
	S. 42.	—
	S. 43.	—
	S. 45.	—
—— E. 15. 6. ——	S. 47.	—
—— E. 16. and 17. 6.	S. 48.	—
	S. 49.	—
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	S. 59.	—
	S. 60.	—
	S. 61.	—
—— E. 18. 6. ——	S. 62.	—
—— E. 20. 6. ——	S. 63.	—
	S. 64.	—
	S. 65.	—
	S. 66.	—
—— E. 22. 6. ——	S. 68.	—
	S. 69.	—
—— E. 23. 6. ——	S. 70.	—
—— E. 24. 6. ——	S. 71.	—
—— E. 30. 6. ——	S. 72.	—
	S. 73.	—
	S. 74.	—
	S. 75.	—

After	E. 31. 6.	may be read	S. 76.	6.
			S. 77.	—
			S. 78.	—
——	E. 33. 6.	——	S. 79.	—
			S. 80.	—
			S. 81.	—
			S. 83.	—
			S. 84.	—
——	E. 2. 12.	——	S. 85.	—
			S. 86.	—
			S. 87.	—
			S. 88.	—
			S. 89.	—
			S. 90.	—

AN EXPLANATION

OF THE SYMBOLS EMPLOYED IN THIS TREATISE, AS ABBREVIATIONS.

=	denotes	*is equal to* or *equal to.*
>	………	*is greater than.*
<	………	*is less than.*
+	………	*together with.*
−	………	*diminished by.*
⊥	………	*perpendicular.*
∠	………	*angle.*
∠∠	………	*angles.*
AB, or $\overline{AB}$	……	*a straight line, of which the points denoted by* A *and* B *are the extremities.*
$\overset{\frown}{AB}$	………	*a circular arch, of which the points denoted by* A *and* B *are the extremities.*
$\overline{AB}^2$	………	*a square, having* $\overline{AB}$ *for one of its sides.*
$\overline{AB}\times\overline{CD}$	……	*a rectangle, of which* $\overline{AB}$ *and* $\overline{CD}$ *are adjacent sides.*
$2\overline{AB}$, &c.	……	*the double, &c. of* $\overline{AB}$.
△	denotes	*a triangle.*
△△	………	*triangles.*
▭	………	*a parallelogram.*
▭▭	………	*parallelograms.*
A:B	………	*the ratio of* A *to* B.
A:B::C:D	…	*the ratio of* A *to* B *is equivalent to the ratio of* C *to* D.
∴	………	*therefore.*

A

SUPPLEMENT

TO THE

ELEMENTS OF EUCLID.

BOOK I.

PROP. I.

1. PROBLEM. *A GIVEN plane rectilineal angle being divided into any number of equal angles, to divide the half of it into the same number of angles, all equal to one another.*

Bisect (E.* 9. 1.) the given angle: And, first, if it be divided into an *odd* number of equal parts, it is evident that the middle part is thereby bisected. Bisect, therefore, each of the remaining equal parts, on either side of that middle part, and the half of the given angle will, manifestly, be divided into as many equal parts as the given angle itself.

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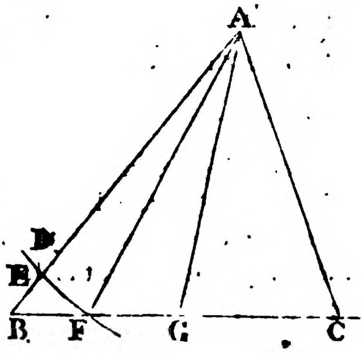
* In this and the following references, the letter E is used to indicate *Euclid's Elements;* the letter S, in like manner, refers to this *Supplement;* the former of the subsequent numbers points out the *Proposition,* and the latter the *Book,* intended to be quoted.

Again, if the given angle be divided into an *even* number of equal parts, it is plain that the straight line which bisects it, will have the half of that number of equal parts, on each side of it. Bisect, therefore, each of the equal parts, on either side of that line; and the half of the given angle will thereby be divided, as before, into as many equal parts as the given angle itself.

PROP. II.

2. PROBLEM. *From the vertex of a given scalene triangle, to draw, to the base, a straight line which shall exceed the less of the two sides, as much as it is itself exceeded by the greater.*

Let ABC be the given scalene triangle, and let AB be greater than AC: It is required to draw,



from the vertex A, to the base BC, a straight line which shall exceed AC, as much as it is exceeded by AB.

From AB cut off (E. 3. 1.) AD = AC; bisect (E. 10. 1.) DB in E; from the centre A, at the distance AE, describe (E. 3. *Post.*) the circle EF cutting BC in F; and join (E. 1. *Post.*) A, F: Then is AF the straight line which was to be drawn.

For, (E. 15. def. 1.) AF = AE; and (*constr.*) AD = AC; ∴ AF − AC = AE − AD = DE.

Also, AB − AE = BE; *i. e.* AB − AF = BE: and (*constr.*) BE = DE.

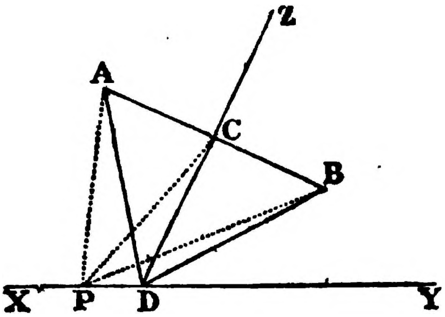
∴ AF − AC = AB − AF.

PROP. III.

3. PROBLEM. *In a straight line given in position, but indefinite in length, to find a point, which shall be equidistant from each of two given points, either on contrary sides, or both on the same side of the given line, and in the same plane with it; but not situated in a perpendicular to it.*

Let XY be a given straight line indefinite in length, and A, B, two given points without it; not situated in a perpendicular to XY: It is required to find a point in XY that shall be equidistant from A and B.

First, let A, B be both on the same side of XY:



Join A, B; bisect (E. 10. 1.) AB in C; from C draw (E. 11. 1.) CD $\perp$ to AB, meeting XY in D. The point D is equidistant from A, B.

For, join A, D and B, D. Then, since (*constr.*) AC = BC, and CD is common to the two △ ACD, BCD, and that (*constr.* and E. 10. def. 1.) ∠ ACD = ∠ BCD, ∴ (E. 4. 1.) AD = BD; *i. e.* D is equidistant from A and B.

But, if the two given points, A and B, are on contrary sides of XY, let them be joined, as before, and let the straight line which joins them be bisected.

Then, if the point of bisection be in XY, that, which was required, has been done. But, if that point be not in XY, draw from it, as before, a perpendicular to AB, and it may be shown, as in the first case, that the point, in which the perpendicular meets XY, is that which was required to be found.

4. COR. 1. By the help of this problem, it is manifest that a circle may be described, which shall have its centre in a given straight line, and

which shall pass through two given points without that line.

5. Cor. 2. It is evident from the demonstration, that any point in an indefinite straight line DZ, which bisects the given finite straight line AB, at right angles, is equidistant from the extremities A and B, of that given finite line: And, any point which is not in that indefinite line DZ, is not equidistant from the two extremities A and B of the given finite line.

For, let P be any point, not in DZ, which bisects AB at right ∠∠ in C; and, if it be possible, let P be equidistant from A and B: Join P, A and P,C and P,B; and since (*hyp.*) AC=CB, and CP is common to the two △△ ACP, BCP, and that (*hyp.*) PA=PB, ∴ (E. 8. 1.) the ∠ ACP= ∠ BCP, and ∵ (E. 10. def. 1.) the ∠ ACP is a right ∠; but (*hyp.*) the ∠ ACD is a right ∠; ∴ the ∠ ACP is equal to the ∠ ACD, the less to the greater, which is impossible; ∴ the point P is not equidistant from A and B.

6. Cor. 3. Hence, an indefinite number of circles may be described all of them passing through two given points: And if any number of circles pass, all of them, through the same two given points, their centres are all in the straight line that bisects at right angles the straight line joining the two given points.

7. Cor. 4. Hence, also, a circle may be described which shall pass through two given points, and which shall have its semi-diameter equal to

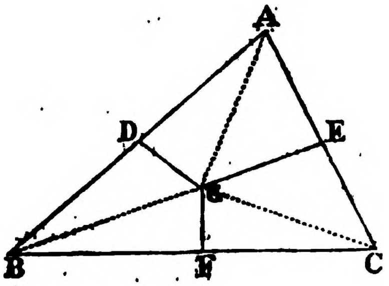
any given finite straight line, that exceeds the half of the straight line joining the two given points.

For, let A, B be the two given points; and join A, B; and let CD be drawn bisecting AB at right ∠∠; from A, as a centre, at a distance equal to the given finite straight line, describe a circle, and let it cut CD in D; ∴ (Cor. 2.) D is equidistant from A and B; and ∴ a circle described from D, as a centre, at the distance DA, which (*constr.* E. 15. def. 1.) is equal to the given semidiameter, will pass through B.

Prop. IV.

8. **Theorem.** *If the three sides of a given triangle be bisected, the perpendiculars drawn to the sides, from the three several bisections, shall all meet in the same point: And that point is equidistant from the three angular points of the given triangle.*

Let ABC be a given △, of which the three sides AB, AC, and CB are bisected in the points D, E



and F, respectively: The perpendiculars drawn to the several sides from D, E, F, shall all meet in a point that is equidistant from A, B and C.

For, draw (E. 11. 1.) $DG \perp$ to AB, and $EG \perp$ to AC, and let them meet in G: Join G, F. Then, (*constr.* and S. 3. 1. Cor. 2. ∴) $BG = AG$, and $AG = CG$; ∴ $CG = BG$.

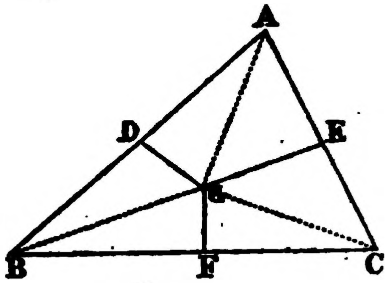
Again, since (*hyp.*) $BF = CF$ (*constr.*) and FG is common to the two $\triangle$s BFG, CFG, and that $BG = CG$, ∴ the $\angle BFG = \angle CFG$ (E. 8. 1.); *i.e.* (E. 10. def. 1.) GF is $\perp$ to BC: And there cannot (E. 10. def. 1.) be drawn from F more than one straight line $\perp$ to BC. It is plain, therefore, that the perpendiculars drawn to the sides, from D, E and F, all meet in the same point G: And, since it has been shown that $AG = BG = CG$, the point G is equidistant from A, B and C.

PROP. V.

9. PROBLEM. *To find a point, in a given plane, which shall be equidistant from three given points in the plane, that are not all in the same straight line.*

Let A, B, C. be three given points, not all of them in the same straight line: It is required to find a point, that shall be equidistant from A, B and C.

Join A, B, and B, C, and C, A; bisect (E. 10. 1.) AB in D, and AC in E; draw (E. 11. 1.) from D



and E, DG ⊥ to AB, and EG ⊥ to AC, and let them meet in G.

Then, (S. 3. 1. Cor. 2.) the point G is equidistant from A, B, and C.

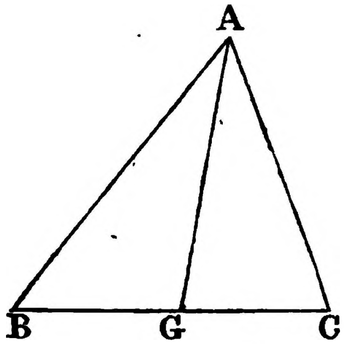
10. Cor. By the help of this problem a circle may be described about a given triangle; or so as that its circumference shall pass through any three given points that are not in the same straight line.

Prop. VI.

11. Theorem. *There cannot be drawn more than two equal straight lines, to another straight line, from a given point without it.*

Let A be a given point, without the given straight line BC: There cannot be drawn more than two equal straight lines, from A to BC.

For, if it be possible, let $\overline{AB}=\overline{AG}=\overline{AC}$; ∴ (E. 5. 1.) $\angle ACB = \angle AGC$: Also $\angle ACB = \angle ABC$; ∴ $\angle AGC = \angle ABC$; *i.e.* the exterior is equal to the interior opposite ∠, when the side



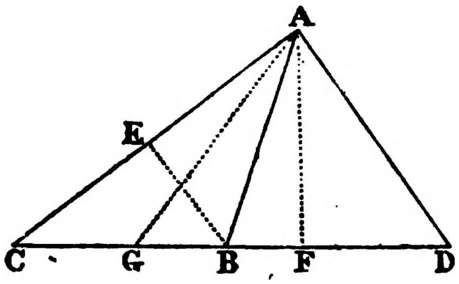
BG, of the △ AGB, is produced: which (E. 16. 1.) is absurd.

12. Cor. A circle cannot cut a straight line in more points than two.

Prop. VII.

13. Theorem. *The perpendicular let fall from the obtuse angle of an obtuse-angled triangle, or from any angle of an acute-angled triangle, upon the opposite side, falls within that side: But the perpendicular drawn to either of the sides containing the obtuse angle of an obtuse-angled triangle, from the angle opposite, falls without that side.*

Let ABC be an obtuse-angled △, obtuse-angled at B, and let ABD be an acute-angled △: The perpendicular drawn from B to AC falls within AC; the perpendicular drawn from any other ∠ A, of the △ ABC, to the opposite side BC, falls without BC; and the perpendicular drawn from



any ∠ A, of the △ ABD, to the opposite side BD, falls within BD.

For, first, if it be possible, let AG, drawn (E. 12. 1.) from A ⊥ to BD, meet DB, produced, in G: Then, since (*hyp.*) the ∠ ABD is acute, the ∠ ABD is (E. 13. 1.) obtuse; and (*constr.*) the ∠ AGB is a right angle: Wherefore the two ∠∠ ABG, AGB of the △ ABG are not less than two right angles; which (E. 17. 1.) is absurd. Therefore, the perpendicular drawn from A on BD cannot fall without BD. And, in the same manner, it may be shewn, that the perpendicular drawn from B on the opposite side AC, of the obtuse-angled △ ABC, cannot fall without AC, and also that the perpendicular drawn from A, on the opposite side BC, of that △, cannot fall within BC.

PROP. VIII.

14. **THEOREM.** *If a straight line, meeting two other straight lines, makes the two interior angles*

on the same side of it not less than two right angles, these lines shall never meet on that side, if produced ever so far.

For, if it be possible, let two straight lines meet, which make, with another straight line, the two interior angles, on the same side, not less than two right ∠ : Then it is plain, that the three straight lines will thus include a △, two ∠ of which are not less than two right angles; which (E. 17. 1.) is absurd. Wherefore, the two straight lines cannot meet, on that side of the straight line, on which they make the two interior ∠ not less than two right ∠.

15. Cor. Two straight lines, which are both perpendicular to the same straight line, are parallel to each other.

Prop. IX.

16. Theorem. *The three sides of a triangle taken together, exceed the double of any one side, and are less than the double of any two sides.*

For, since (E. 20. 1.) any two sides of a △ are > the third, if the third side be added both to those two and to itself; it is evident that the three sides are, together, > the double of the third.

Again, since (E. 20. 1.) any side of a △ is < the other two, if the other two be added both to that side, and to themselves, it is evident, that the three sides are, together, < than the double of the other two.

PROP. X.

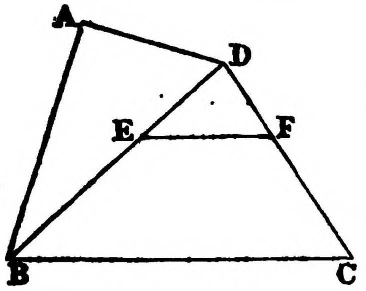
17. THEOREM. *Any side of a triangle is greater than the difference between the other two sides.*

If, the △ be equilateral, or isosceles, the proposition is manifestly true. But let it be a scalene △: Then, since (E. 20. 1.) any two sides of the △ are > the third, if either of those two be taken from that third side, it is plain that the remaining side is greater than the difference of the other two.

PROP. XI.

18. THEOREM. *Any one side of a rectilineal figure is less than the aggregate of the remaining sides.*

Let ABCD be a given rectilineal figure: Any one side, as BC, is less than the aggregate of the remaining sides.



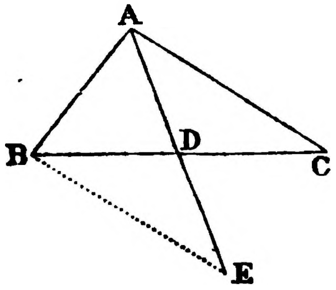
For, first, let the figure be quadrilateral; and join B,D: Then (E. 20. 1.) $BD + DC > BC$; and, $BA + AD > BD$; $\therefore BA + AD + DC > BD + DC$; much more, then, is $BA + AD + DC > BC$.

And the proposition may, in the same manner, be proved to be true, when the figure has more than four sides.

PROP. XII.

19. THEOREM. *The two sides of a triangle are together, greater than the double of the straight line which joins the vertex and the bisection of the base.*

Let ABC be any given △, and let AD be the straight line joining the vertex A, and the bisection, D, of the base BC: $AB + AC > 2AD$. Produce AD to E, and cut off (E. 3. 1.) $DE = AD$; also, join B, E.

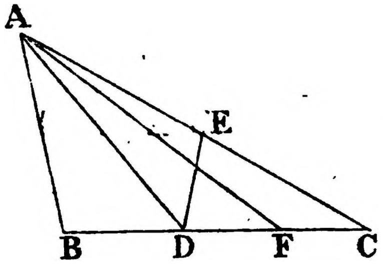


Then since (*hyp.*) $BD = DC$, and (*constr.*) $AD = DE$, the two sides BD, DE, of the △ BDE, are equal to the two sides AD, DC of the △ ADC;

and (E. 15. 1.) the ∠ BDE = ∠ ADC; ∴ (E. 4. 1.) BE = AC. But (E. 20. 1.) AB + BE > AE; but AC has been proved to be equal to BE, and AE is (*constr.*) the double of AD; ∴ AB + AC > 2 AD.

Prop. XIII.

20. Theorem. *The two sides of a triangle are, together, greater than the double of the straight line drawn from the vertex to the base, bisecting the vertical angle.*



Let ABC be any given △, and let AD be drawn from the vertex A, to the base BC, bisecting the vertical ∠ BAC: Then, AB + AC > 2 AD.

If the given △ be isosceles, the straight line which bisects the vertical ∠ is (E. 4. 1.) ⊥ to the base; and since (E. 17. 1. and E. 19. 1.) each of the equal sides is greater than the perpendicular, the proposition, is, in this case, manifestly true.

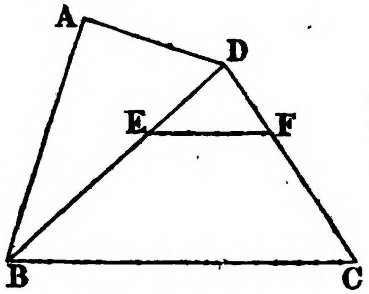
But, let ABC be a scalene △, and let the side AB be less than AC: Then, of the segments into which AD, bisecting the ∠ BAC, divides the base BC, BD, which is adjacent to the less side AB, is the less.

For, from AC, the greater, cut off (E. 3. 1.) $AE = AB$, the less, and join D, E; and because BA, AD are equal to EA, AD, and (*hyp.*) the $\angle BAD = \angle EAD$, $\therefore$ (E. 4. 1.) $BD = DE$, and $\angle BDA = \angle EDA$; but (E. 16. 1.) $\angle DEC > \angle ADE$; $\therefore$ $\angle DEC > \angle ADB$; and (E. 16. 1.) $\angle ADB > \angle ACD$; much more then is $\angle DEC > \angle ECD$; $\therefore$ (E. 19. 1.) $DC > DE$; but it has been shewn that $DE = DB$; $\therefore$ $DC > DB$. From DC, the greater cut off (E. 3. 1.) $DF = DB$; and join A, F: Then (E. 16. 1.) the $\angle AFC > \angle ABC$; and because (*hyp.*) $AC > AB$, $\therefore$ (E. 18. 1.) $\angle ABC > \angle ACB$; much more then is $\angle AFC > \angle ACF$; $\therefore$ (E. 19. 1.) $AC > AF$: But (S. 12. 1. and *constr.*) $AB + AF > 2AD$; much more then is $AB + AC > 2\,AD$.

21. Cor. From the demonstration it is manifest, that of the segments into which the straight line bisecting the vertical $\angle$ of a scalene $\triangle$, divides the base, that which is adjacent to the less side, is the less.

Prop. XIV.

22. Theorem. *If a trapezium and a triangle stand upon the same base, and on the same side of it, and the one figure fall within the other, that which has the greater surface shall have the greater perimeter.*



Let the trapezium EBCF fall within the △ DBC; let, also, the △ DBC fall within the trapezium ABCD; and let all the figures stand on the same base BC: The perimeter of the △ DBC is > the perimeter of EBCF, and < the perimeter of ABCD.

First, let E and F be in the sides DB and DC of the △ DBC, and let the vertex D of the △ DBC coincide with the ∠ A or the ∠ D of the trapezium ABCD.

Then, since (E. 20. 1.) DE+DF>EF, add to both, EB, BC, and CF; ∴ DE+EB+DF+FC+BC>EF+FC+CB+BE; *i. e.* the perimeter of the △ DBC > the perimeter of EBCF.

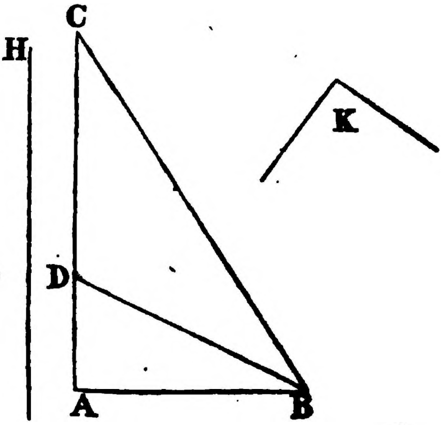
Again, since (E. 20, 1.) BA+AD>BD, add to both DC and CB; ∴ BA+AD+DC+CB> BD+DC+CB; *i. e.* the perimeter of the trapezium ABCD > the perimeter of the △ DBC.

And, if E or F fall within the △ DBC, and the vertex of the △ do not coincide with either of the ∠∠ A or D, of the trapezium, it may, in the same manner, be proved, that the proposition is true, a fortiori.

PROP. XV.

23. PROBLEM. *One of the angles at the base of a triangle, the base itself, and the aggregate of the two remaining sides, being given, to construct the triangle.*

Let K be the given angle, AB the given base of the triangle, and H the aggregate of the two remaining sides: It is required to construct the triangle.



At the point A, in AB, make (E. 23. 1.) the ∠ BAC = ∠ K, and make (E. 3. 1.) AC = H; join C, B; and at the point B, in CB, make (E. 23. 1.) the ∠ CBD = ∠ ACB: Then is DAB the triangle which was to be constructed.

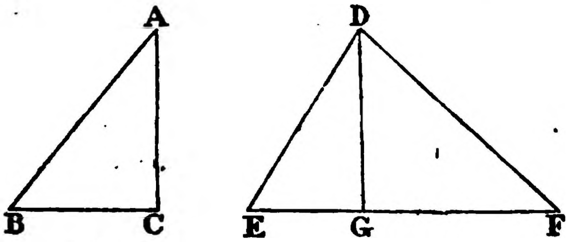
For, since (*constr.*) ∠ DCB = ∠ DBC, ∴

(E. 6. 1.) BD = DC; add to both DA; ∴ BD + DA = CD + DA; *i. e.* BD + DA = CA; and (*constr.*) CA = H; ∴ BD + DA = H; and the ∠ A was made equal to the given ∠ K: It is manifest, therefore, that DAB is the triangle which was to be constructed.

Prop. XVI.

24. Theorem. *If two right-angled triangles have the three angles of the one equal to the three angles of the other, each to each, and if a side of the one be equal to the perpendicular let fall from the right angle upon the hypotenuse of the other, then shall a side of this latter triangle be equal to the hypotenuse of the former.*

Let ACB and EDF be two right angled △s,



right angled at C and D, having, also the ∠ DEF = ∠ ABC, the ∠ EFD = ∠ CAB, and the side AC, of the △ ABC, equal to the perpendicular DG, drawn D to the hypotenuse EF of the △ DEF: The side DE, of the △

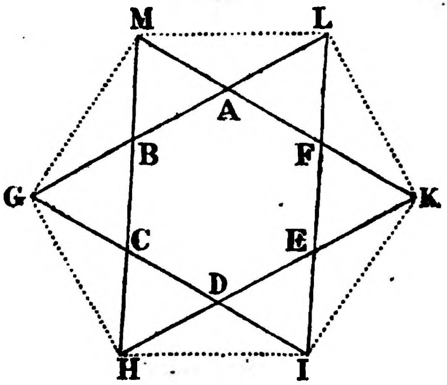
DEF, is equal to the hypotenuse AB, of the △ ABC.

For, since AC = DG, and the two ∠∠ ACB, ABC, of the △ ABC, are equal to the two ∠∠, DGE, DEG, of the △ DEG, each to each, ∴ (E. 26. 1.) DE = AB.

PROP. XVII.

25. **THEOREM.** *If the sides of any given equilateral and equiangular figure of more than four sides, be produced so as to meet, the straight lines, joining their several intersections, shall contain an equilateral and equiangular figure, of the same number of sides as the given figure.*

Let ABCDEF be any equilateral and equiangular figure, of more than four sides; let the sides, produced, meet in the points G, H, I, K, L, M; and let those points of intersection be



joined: Then is GHIKLM an equilateral and equiangular figure, of the same number of sides as ABCDEF.

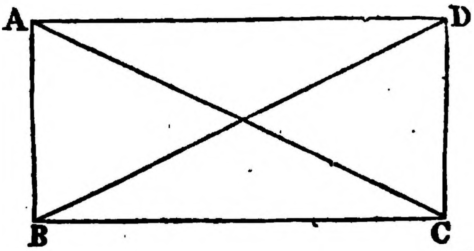
For, since (*hyp.*) the ∠ A, B, C, D, E, F are all equal, the △MAB, GBC, HCD, IDE, KEF, LFA are all (E. 13. 1. and E. 6. 1.) isosceles, and any two of them have their ∠ equal, each to each; ∴ since (*hyp.*) BA = AF, and that the ∠ MAB, MBA are equal to the ∠ LFA, LAF, each to each, the side MA of the △ MAB = the side LA (E. 26. 1.) of the △ LAF; and in the same manner it may be shewn that MB = GB, GC = CH, HD = DI, IE = EK, and KF = FL: But, because the ∠ of the figure ABCDEF are (*hyp.*) equal, ∴ (E. 15. 1.) the ∠ LAM, MBG, GCH, HDI, IEK, KFL, are all equal to one another; ∴ (E. 4. 1.) the sides LM, MG, GH, HI, IK and KL are all equal, as are also the ∠ of the △LAM, MBG, GCH, HDI, IEK, and KFL, each to each: And the ∠ AMB, BGC, CHD, DIE, EKF, and FLA have been shewn to be equal to one another: Wherefore the figure GHIKLM is equilateral and equiangular; and it is manifest that it has the same number of sides as the figure ABCDEF.

PROP. XVIII.

26. THEOREM. *If two opposite sides of a quadrilateral figure be equal to one another, and the two*

remaining sides be also equal to one another, the figure is a parallelogram.

Let any two opposite sides, as AB, DC, of



the quadrilateral figure ABCD, be equal to one another, and let the two remaining sides, AD, BC, be, also, equal to one another: The figure ABCD is a ▭.

For, join D, B: Then since the two sides AD, DB, of the △ ADB, are equal to the two sides CB, BD, of the △ CBD, and that the base AB is equal (*hyp.*) to the base DC, ∴ (E. 8. 1.) the ∠ ADB = ∠ DBC; and (E. 4. 1.) the ∠ ABD = ∠ BDC; ∴ (E. 27. 1.) AD is parallel to BC, and AB is parallel to DC; *i. e.* the figure ABCD is a ▭.

27. Cor. 1. Hence may be deduced a practical method of drawing a straight line, through a given point, parallel to a given straight line.

For, let it be required to draw through the given point B, a straight line parallel to AD: From any point A in AD, as a centre, and at any distance, describe a circle cutting AD in D; and from B as a centre, at the same distance, describe another circle; lastly, from D as a centre, at a distance equal to that of A, B, describe another circle, cutting the circle last described in C; join B, C. BC is parallel to AD.

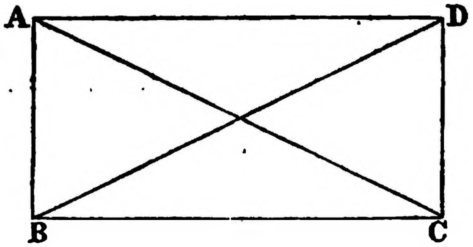
For, if A, B and D, C be joined, it is manifest from the construction, that AD = BC, and AB = DC: ∴ (S. 16. 1.) BC is parallel to AD.

28. Cor. 2. A rhombus is a parallelogram.

Prop. XIX.

29. Theorem. *Every parallelogram which has one angle a right angle, has all its angles right angles.*

Let one ∠, as A, of the ▭ABCD be a right angle: The ∠∠ B, C, and D are also right angles.

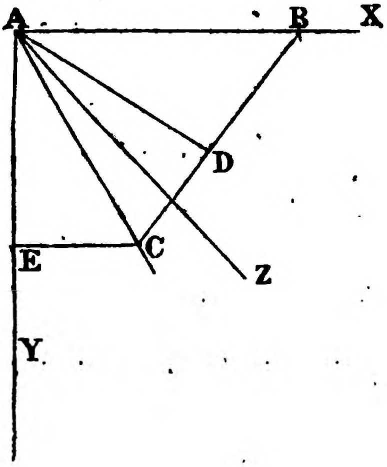


For, since AD is parallel to BC, and AB meets them, the two interior ∠∠ A, B are, (E. 29. 1.) together, equal to two right ∠∠; but (*hyp.*) the ∠A is a right ∠; ∴ the ∠B is also a right ∠: And, in the same manner, may the remaining ∠∠, C and D, be shewn to be right ∠∠.

PROP. XX.

30. **PROBLEM.** *To trisect a right angle; i. e. to divide it into three equal parts.*

Let the ∠XAY be a right ∠: It is required



to trisect it; *i. e.* to divide it into three equal parts.

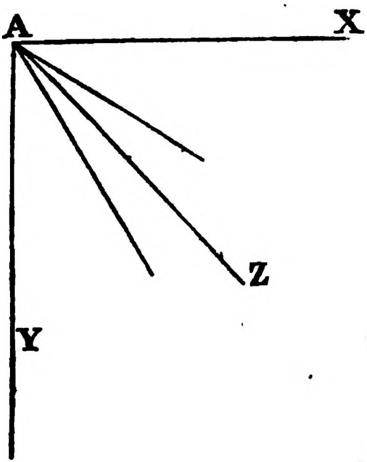
In AX take any point B; upon AB describe (E. 1. 1.) the equilateral △ ACB; and from A draw (E. 12. 1.) AD ⊥ to BC: The ∠XAY is trisected by the two straight lines AC and AD.

For, from C draw (E. 12. 1.) draw CE ⊥ to AY; then, since the ∠∠ BAE, AEC, are right ∠∠ ∴ (E. 28. 1.) AB is parallel to EC; ∴ (E. 29. 1.) ∠ECA = ∠CAB = ∠ACB; because (*constr.*) the △ ACB is equilateral, and (E. 5. 1. *cor.*) equiangular: Since, ∴, the ∠ ACE = ∠ACD, and that the ∠∠ D and E are right ∠∠, and AC is common to the two △△ ADC, AEC, ∴ (E. 26. 1.) the ∠EAC=∠DAC: Again, since (*constr.* and E. 5. 1. *cor.*) the ∠ACB=∠ABC, and (*constr.*) the ∠∠ at D are right angles, and that AC=AB, ∴ (E. 26. 1.) the ∠DAC=∠DAB: But it was shewn that the ∠EAC=∠DAC: ∴ ∠EAC=∠DAC=∠DAB; *i. e.* the right ∠XAY is trisected by AC and AD.

PROP. XXI.

31. PROBLEM. *Hence, to trisect a given rectilineal angle, which is the half, or the quarter, or the eighth part, and so on, of a right angle.*

First, let the given ∠YAZ, be the half of a



right ∠, and let it be required to trisect it.

Draw (E. 11. 1.) from A, $\overline{AX} \perp \overline{AY}$; trisect (S. 18. 1.) the right ∠XAY; then (S. 1. 1.) trisect the ∠YAZ, which is the half of the ∠YAX.

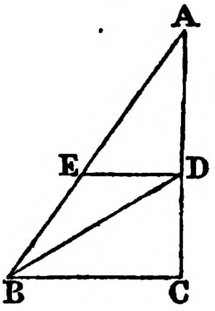
But, if the given ∠ be the quarter of a right angle, its double may be trisected by the former case; and ∴ the given ∠ itself may be trisected by (S. 1. 1.)

And, by following the same method, it is evident that an ∠ may be trisected, which is the eighth part, or the sixteenth part, and so on, of a right angle.

PROP. XXII.

32. **PROBLEM.** *In the hypotenuse of a right-angled triangle, to find a point, the perpendicular distance of which from one of the sides, shall be equal to the segment of the hypotenuse between the point and the other side.*

Let ABC be a right-angled △, right-angled



at C: It is required to find a point in the hypotenuse AB, the perpendicular distance of which from one of the sides, as AC, shall be equal to the segment of the hypotenuse between that point, and BC.

Bisect (E. 9. 1.) the $\angle$ ABC, by $\overline{BD}$, and let BD meet AC in D; through D, draw $\overline{DE}$ (E. 31. 1.) parallel to CB: E is the point which was to be found.

For, since DE is parallel to CB, the $\angle CBD = \angle BDE$ (E. 29. 1.); but (*constr.*) the $\angle CBD = \angle DBE$; $\therefore \angle DBE = \angle BDE$; $\therefore$ (E. 6. 1.) $ED = EB$; and since (*hyp.*) the $\angle$ C is a right $\angle$, and that DE is parallel to CB, the $\angle$ CDE (E. 29. 1.) is a right $\angle$; *i.e.* ED is $\perp$ to AC.

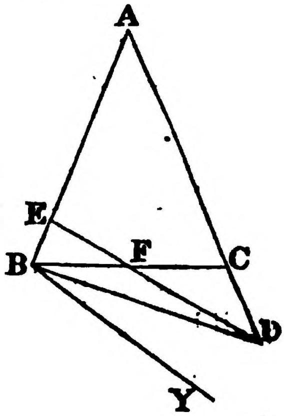
PROP. XXIII.

83. **PROBLEM.** *In the base of a given acute-angled triangle, to find a point, through which if a straight line be drawn perpendicular to one of the sides, the segment of the base, between that side and the point, shall be equal to the segment of the perpendicular, between the point and the other side produced.*

Let ABC be the given acute-angled $\triangle$: It is required, to find, in the base BC, a point through which if a perpendicular be drawn to AB, the segment of the base, between that point and the point

B, shall be equal to the segment of the perpendicular between that same point and AC produced.

Draw (E. 11. 1.) from B, $\overline{\text{BY}}$ ⊥ to $\overline{\text{AB}}$; bisect (E. 9. 1.) the ∠ CBY by $\overline{\text{BD}}$, meeting AC, produced in D; through D, draw (E. 31. 1.) $\overline{\text{DE}}$ parallel to BY, and let DE cut BC in F: F is the point which was to be found.

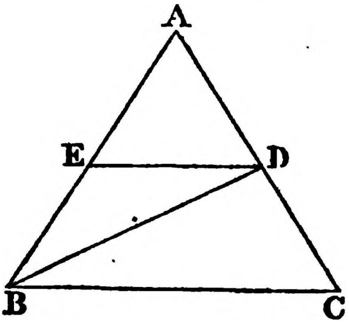


For, since (*constr.*) the ∠ ABY is a right ∠, and that DE is parallel to BY, the ∠ E (E. 29. 1.) is, also, a right ∠; and the ∠ YBD = ∠ BDF; but (*constr.*) the ∠ YBD = ∠ DBF; ∴ the ∠ DBF = ∠ BDF; ∴ (E. 6. 1.) FB = FD.

PROP. XXIV.

34. PROBLEM. *From a given isosceles triangle to cut off a trapezium, which shall have the same base as the triangle, and shall have its three remaining sides equal to each other.*

Let ABC be the given isosceles △: It is required to cut off from it a trapezium, which, having BC for its base, shall have its three remaining sides equal to one another.



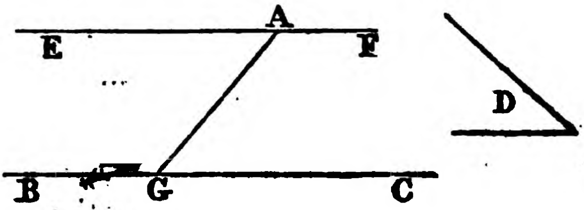
Bisect (E. 9. 1.) the ∠ ABC by BD, meeting AC in D; and through D draw (E. 31. 1.) DE parallel to CB: Then shall BE, ED, and DC, the three sides of the trapezium BEDC, be equal to one another.

For, since DE is parallel to BC, the ∠ AED = ∠ ABC (E. 29. 1.), and ∠ ADE = ∠ ACB; but (*hyp.* and E. 5. 1.) ∠ ABC = ∠ ACB; ∴, ∠ AED = ∠ ADE; ∴ (E. 6. 1.) AE = AD; but (*hyp.*) AB = AC; from these equals take the equals AE and AD, there remains EB = DC: Again, because DE is parallel to BC, the ∠ CBD = ∠ BDE (E. 29. 1.); but (*constr.*) ∠ CBD = ∠ DBE; ∴ the ∠ DBE = ∠ BDE; ∴ (E. 6. 1.) EB = ED; and EB has been proved to be equal to DC; ∴ EB, ED and DC are equal to one another.

PROP. XXV.

35. **PROBLEM.** *To draw to a given straight line, from a given point without it, another straight line which shall make with it an angle equal to a given rectilineal angle.*

Let BC be a given straight line, A a given point



without it, and D a given rectilineal ∠: It is required to draw from A, a straight line which shall make with [illegible] an ∠ equal to the ∠ D.

Through A draw (E. 31. 1.) EAF parallel to BC; at the point A in EAF, make (E. 23. 1.) the ∠ EAG = ∠ D : AG is the line which was to be drawn.

For, since (*constr.*) EF is parallel to BC, the ∠ EAG = ∠ AGC (E. 29. 1.); but (*constr.*) the ∠ EAG = ∠ D; ∴ ∠ AGC = ∠ D.

PROP. XXVI.

36. **THEOREM.** *If all the angles but one of any rectilineal figure, be together, equal to all the*

angles but one, of another rectilineal figure having the same number of sides, the remaining angle of the one figure, shall be equal to the remaining angle of the other: And, conversely, if an angle in the one figure be equal to an angle in the other, the remaining angles of the one shall be equal, together, to the remaining angles of the other.

For, since the two figures have the same number of sides, all the interior ∠∠ of the one are, together, equal (E. 32. 1. cor. 1.) to all the interior ∠∠ of the other: If ∴ from these equals be taken first, the aggregates which, by the hypothesis, are equal; and secondly, the single angles, which are supposed to be equal, it is manifest that the remaining angle, or angles, of the one figure, must be equal to the remaining angle, or angles of the other.

Prop. XXVII.

37. Theorem. *The angle at the base of an isosceles triangle is equal to, or is less, or greater, than the half of the vertical angle, accordingly as the triangle is a right-angled, an obtuse-angled, or an acute-angled triangle.*

For, (E. 5. 1, and E. 32. 1.) the double of the

∠ at the base + the vertical ∠ = two right ∠∠; if ∴ the vertical ∠ be a right ∠, and if it be taken from both, there remains the double of the ∠ at the base = a right ∠; ∴ the ∠ at the base = half of a right ∠.

But, if the vertical ∠ be obtuse, when it is taken away from the same equals as before, there will remain the double of the ∠ at the base equal to a less ∠ than a right ∠; ∴ the ∠ at the base is, in this case, less than the half of a right ∠.

And, in like manner it may be shewn, that, when the vertical ∠ of the isosceles △ is acute, the ∠ at the base is greater than the half of a right angle.

Prop. XXVIII.

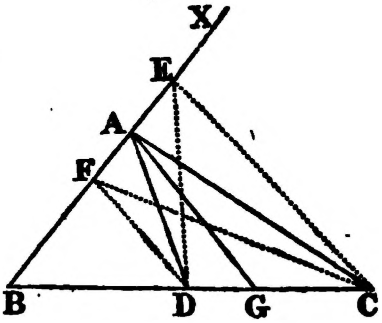
38. Theorem. *If either of the equal sides of an isosceles triangle be produced, towards the vertex, the straight line, which bisects the exterior angle, shall be parallel to the base.*

For, (E. 5. 1. and E. 32. 1.) the exterior ∠ at the vertex of an isosceles △ is the double of either of the ∠∠ at the base; ∴ the half of that interior ∠ is equal to either of the ∠∠ at the base; ∴ the straight line bisecting the vertical ∠ is (E. 28. 1.) parallel to the base.

PROP. XXIX.

89. **THEOREM.** *The distance of the vertex of a triangle from the bisection of its base, is equal to, greater than, or less than the half of the base, accordingly as the vertical angle is a right, an acute, or an obtuse angle.*

First, let ABC be a right-angled $\triangle$, right-



angled at A, and let AD join A and the bisection, D, of the base: $AD = DB$, or DC.

For, if not, AD is either greater or less than BD: Produce BA to X; and first, if it be possible, let $AD > DB$; $\therefore$, also, $AD > DC$; $\therefore$ (E. 18. 1.) $\angle B > \angle BAD$, and $\angle C > \angle CAD$; $\therefore \angle B + \angle C > \angle BAD + \angle CAD$; *i. e.* $\angle B + \angle C > \angle BAC$; but (*hyp.* and E. def. 10. 1.) $\angle BAC = CAX$; $\therefore \angle B + \angle C > \angle CAX$; which (E. 32. 1.) is absurd.

And, in like manner, if DA be supposed to be less than BD, it may be shewn that $\angle B + \angle C <$

∠ CAX; which is absurd. Therefore, DA = DB, or DC.

Next, let the vertical ∠ CEB, of the △ EBC, be acute, and let ED join E and the bisection, D, of BC, ED > BD, or DC.

From either of the ∠∠ B or C, as C, if the △ EBC be acute-angled, draw (E. 12. 1.) CA ⊥ to the opposite side EB; and join A, D: Then (S. 7. 1.) CA falls within EB; and, since (*constr.*) the ∠ CAE is a right ∠, the ∠ DAE is greater than a right ∠; ∴ (E. 17. 1.) the ∠ AED is less than a right ∠, and ∴ less than the ∠ DAE; ∴ (E. 19. 1.) DE > DA; but, by the former case, DA = DB; ∴ DE > DB, or DC.

Lastly, if FBC be an obtuse-angled △, obtuse-angled at F, join F, D; draw, as before, CA ⊥ BF; and join A, D: Then (S. 7. 1.) CA falls without BF, and the ∠ AFD (E. 16. 1.) > the ∠ FBD; but since (1st case) DA = DB, the ∠ DBF = ∠ DAF (E. 5. 1.); ∴ ∠ AFD > ∠ DAF; ∴ (E. 19. 1.) DA > DF; but DA = DB; ∴ DF < DB, or DC.

Or, the two last cases may be proved, ex absurdo, in the same manner as the first is proved.

40. Cor. 1. If any number of triangles have a right angle for their common vertical angle, and have equal hypotenuses, the locus of the bisections of the several hypotenuses is a quadrantal arch of a circle, having the common vertex for its centre, and the half of any hypotenuse for its radius.

For, the bisections of the hypotenuses will, each of them, (S. 29. 1.) be at a distance from the common vertex equal to the half of one of the equal hypotenuses; *i. e.* they will all be at distances from that point, equal to the half of any one of those equal lines: It is manifest, ∴, that they will be in the circumference of a circle, described from that point as the centre, at a distance equal to the half of one of the hypotenuses.

41. Cor. 2. A circle described from the bisection of the hypotenuse of a right-angled triangle as a centre, at the distance of half the hypotenuse, will pass through the summit of the right angle.

42. Cor. 3. The vertical angle of a △ being a right angle, a point in the base, which is equidistant from the vertex and from either extremity of the base, bisects the base.

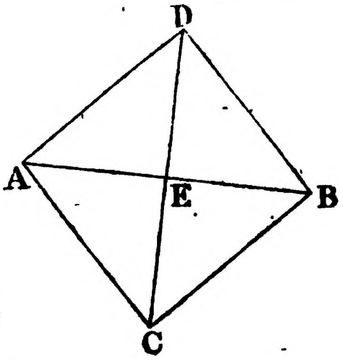
Let the point D, in the base BC of the △ ABC, having the ∠ B a right angle, be equidistant from either extremity, as B, of BC, and from the angular point A: The point D bisects BC.

For, if not, let G be the bisection of BC, and join D, A and E, A: Then, since (*hyp.*) DA = DB, ∴ (E. 5. 1.) the ∠ DAB = ∠ DBA: also, since G is the bisection of BC, ∴ (S. 29. 1.) GA = GB; ∴ (E. 5. 1.) the ∠ GAB = ∠ GBA; ∴ the ∠ GAB = ∠ DAB, the greater to the less, which is absurd; ∴ no other point than D can be the bisection of BC.

PROP. XXX.

43. PROBLEM. *Upon a given finite straight line, as a* diameter, *to describe a square.*

Let AB be a given finite straight line: Upon



AB, as a diameter, it is required to describe a square.

Bisect (E. 10. 1.) AB in E; through E draw (E. 11. 1.) $\overline{\text{DEC}}\perp$ to AB, and make (E. 3. 1.) ED and EC each of them equal to AE or EB: Join A, D, and D, B, and B, C, and C, A: The figure ADBC is a square, having AB for its diameter.

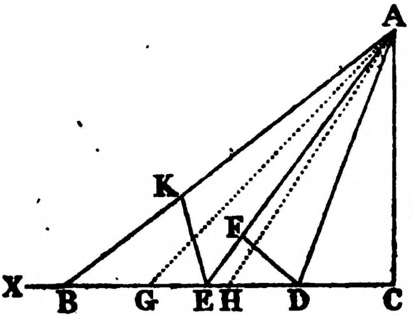
For since (*constr.*) DE = EC, and AE is common to the △ AED, AEC, and that the right ∠ AED = right ∠ AEC, ∴ (E. 4. 1.) AD = AC; and in the same manner AD may be shewn to be equal to DB, and DB to BC; ∴ the figure ADBC is equilateral.

Again, since (*constr.*) AE = DE, the ∠ EAD = ∠ EDA (E. 5. 1.); but (*constr.*) ∠ AED is a right ∠; ∴ each of the ∠∠ EAD, EDA, is half a right ∠; and, in the same manner, may each of the ∠∠ EDB, DBE, CBE, BCE, ECA, EAC, be shewn to be half a right ∠; ∴ all the ∠∠ of the figure ADBC are right ∠∠; and it has been proved that all its sides are equal; ∴ (E. 30. def. 1.) ADBC is a square.

PROP. XXXI.

44. THEOREM. *If either of the acute angles of a given right-angled triangle be divided into any number of equal angles, then, of the segments of the base, subtending those equal angles, the nearest to the right angle is the least; and, of the rest, that which is nearer to the right angle is less than that which is more remote.*

Let ACB be a right-angled △, right-angled at C,



and let the acute ∠ BAC be divided into any number of equal $\angle$s, CAD, DAE, EAB, &c.; then is CD the least of the segments of the base subtending those equal $\angle$s, and of the rest $DE < EB$; and so on.

For, at the point D in AD make (E. 23. 1.) the $\angle ADF = \angle ADC$: And since, also, the $\angle CAD = \angle DAE$ (*hyp.*) and AD common to the two $\triangle$s ACD, AFD, $\therefore$ (E. 26. 1.) $DF = DC$: But (E. 19. 1. and E. 32. 1.) $DE > DF$; $\therefore DE > DC$; *i. e.* $DC < DE$.

Again, at the point E, in AE, make the $\angle AEK = \angle AED$; and it may, in like manner, be shewn that $EK = ED$: But (E. 16. 1.) $\angle BKE > \angle AEK$; $\therefore \angle BKE > \angle AED$; and $\angle AED > \angle ABE$; much more then is $\angle BKE > \angle EBK$; $\therefore$ (E. 19. 1.) $BE > EK$ or ED; *i. e.* $ED < EB$.

And in the same manner may EB be shewn to be less than the next segment that is more remote from C; and so on.

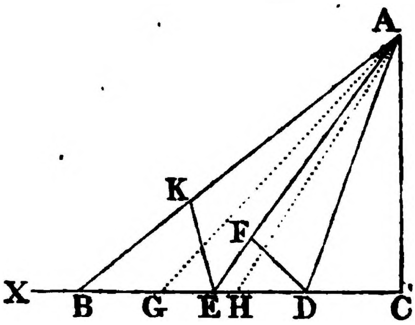
45. Cor. It is manifest, from the demonstration, that if any three straight lines AB, AE, AD, be drawn to the given straight line XC from a given point A, without it, so that the $\angle BAE = \angle EAD$, the segment BE, of XC, which is the further from the perpendicular AC, shall be greater than the segment ED, which is the nearer to AC.

Prop. XXXII.

46. Theorem. *If either angle at the base of a*

triangle be a right angle, and if the base be divided into any number of equal parts, that which is adjacent to the right angle shall subtend the greatest angle at the vertex; and, of the rest, that which is nearer to the right angle shall subtend, at the vertex, a greater angle than that which is more remote.

Let ACB be a right-angled △, right-angled at C, and let the base BC be divided into any number of equal parts CD, DH, HG, &c.: Of these segments DC shall subtend the greatest ∠ at the vertex A; and of the rest DH shall subtend, at A, a greater ∠ than HG; and so on.



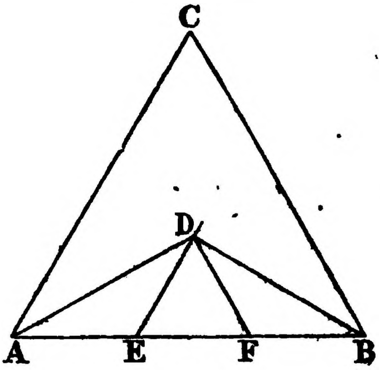
For, join A, D, and A, H, and A, G, &c.; also, at the point A, in DA, make (E. 23. 1.) the $\angle DAE = CAD$: Then (S. 31. 1.) $ED > DC$; but (*hyp.*) $DC = DH$; $\therefore ED > HD$, and it is manifest that the $\angle EAD > \angle HAD$; but (*constr.*) $\angle EAD = \angle CAD$; $\therefore \angle CAD > \angle DAH$:

And, in the same manner, it may be shewn, by the help of the corollary to S. 30. 1. that the $\angle$ DAH > the $\angle$ HAG; and so on.

Prop. XXXIII.

47. **Problem.** *To trisect a given finite straight line.*

Let AB be the given straight line: It is required to divide it into three equal parts.



Upon AB describe (E. 1. 1.) the equilateral $\triangle$ CAB; bisect (E. 9. 1.) the two equal $\angle\angle$ A and B, by the straight lines AD and BD, which meet in D; and from D draw (E. 31. 1.) $\overline{DE}$ parallel to $\overline{CA}$, and $\overline{DF}$ parallel to CB: Then are AE, EF and FB equal to one another.

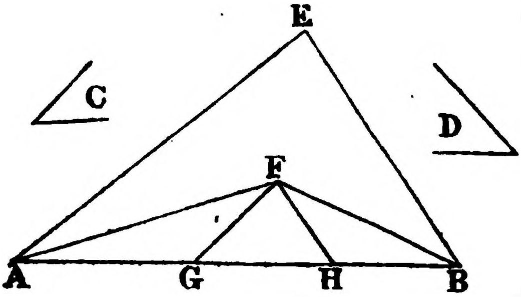
For, since (E. 29. 1. and *constr.*) the $\angle$ DEF = $\angle$ CAB, and $\angle$ DFE = $\angle$ CBA $\therefore$ (S. 26. 1.) $\angle$ EDF = $\angle$ ACB; but (E. 5. 1. *cor.* and *constr.*) the $\triangle$ CAB is equiangular; $\therefore$ the $\triangle$DEF is equiangular; and $\therefore$ (E. 6. 1. *cor.*) it is, also, equilateral; so that DE and DF are, each of them, equal to EF.

Again, since (E. 29. 1. and *constr.*) the ∠EDA = ∠DAC; and that (*constr.*) the ∠DAC = ∠DAE, ∴ ∠EDA = ∠DAE; ∴ (E. 6. 1.) AE = DE; but DE has been proved to be equal to EF; ∴ AE = EF; and in the same manner, EF may be shewn to be equal to FB; ∴ AB has been divided into the three equal parts AE, EF, and FB.

PROP. XXXIV.

48. PROBLEM. *To describe a triangle which shall have its three sides, taken together, equal to a given finite straight line, and its three angles equal to three given angles, each to each; the three given angles being together equal to two right angles.*

Let AB be a given finite straight line, and C and



D two given rectilineal angles: It is required to describe a triangle, which shall have its perimeter equal to AB, two of its angles equal to C and D, each to each, and its third angle equal to an angle, which, together with C and D, makes up two right angles.

At the point A, in AB, make (E. 23. 1.) the ∠ BAE = ∠ C; and at the point B make the ∠ ABE = ∠ D; ∴ (S. 26. 1.) the ∠ AEB is equal to the third ∠ of the △ which is to be described: Bisect (E. 9. 1.) the ∠∠ EAB, EBA, by $\overline{AF}$ and $\overline{BF}$, which meet in F; and through F draw (E. 31. 1.) FG parallel to EA, and FH parallel to EB: Then is FGH the △, which was to be described.

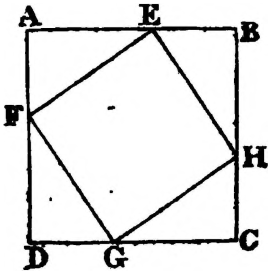
For, since (*constr.*) FG is parallel to EA, and FA meets them, ∴ (E. 29. 1.) the ∠ EAF = ∠ AFG; but (*constr.*) the ∠ EAF = ∠ FAG; ∴ the ∠ FAG = ∠ AFG; ∴ (E. 6. 1.) FG = GA; and, in the same manner, it may be shewn that FH = HB; ∴ FG + GH + HF = AG + GH + HB; *i. e.* the perimeter of the △ FGH is equal to the given straight line AB.

Again, because FG is parallel to EA, and FH is parallel to EB, ∴ (E. 29. 1.) the ∠ FGH = ∠ EAB, and ∠ FHG = ∠ EBA; but (*constr.*) the ∠ EAB = ∠ C, and the ∠ EBA = ∠ D; ∴ also, the ∠ FGH = ∠ C, and the ∠ FHB = ∠ D; ∴ (S. 26. 1.) the ∠ GFH is equal to the third ∠ of the △, which was to be described; ∴ the △ FGH, the perimeter of which has been shewn to be equal to the given straight line AB, is the △ which was to be described.

PROP. XXXV.

49. THEOREM. *If, in the sides of a given square, at equal distances from the four angular points, four other points be taken, one in each side, the figure contained by the straight lines which join them, shall also be a square.*

Let ABCD be a given square; in the sides



AB, BC, CD, DA, let the four points E, H, G, F be taken, so that E is at the same distance from A that H is from B, that G is from C, and F from D; and let E, H, and H, G, and G, F, and F, E, be joined: The figure EFGH is a square.

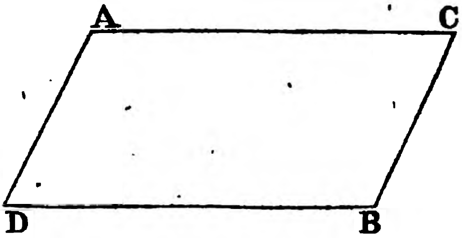
For, since (E. 30. def. 1.) all the sides of the given square ABCD are equal, and that (*hyp.*) AE = BH = DF, it is manifest that the two △s FAE, EBH have the two sides FA, AE equal to the two EB, BH, each to each, and (E. 30. def. 1.) the ∠ A = ∠ B; ∴ (E. 4. 1.) the ∠ AFE = ∠ BEH; and FE = EH: And, in the same manner, it may be shewn that EH = HG = GF; ∴ the figure EFGH is equilateral.

Again, since, as hath been proved, the ∠ AFE = ∠ BEH, ∴ the ∠ AFE + ∠ AEF = ∠ BEH + ∠ AEF; but, since the ∠ A is a right ∠, ∴ (E. 32. 1.) ∠ AFE + ∠ AEF = a right ∠; ∴ also, ∠ BEH + ∠ AEF = a right ∠; but (E. 15. 1. Cor. 2.) ∠ BEH + ∠ AEF + ∠ HEF = two right ∠∠; ∴ the ∠ HEF is a right angle; and, in the same manner, may the remaining ∠∠ of the figure EFGH, which has been shewn to be equilateral, be proved to be right ∠∠; ∴ (E. 30. def. 1.) EFGH is a square.

Prop. XXXVI.

50. **Theorem.** *If the opposite angles, of a quadrilateral figure be equal to each other, the figure shall be a parallelogram.*

Let AB be a quadrilateral figure, having the angle A equal to the opposite angle B, and the angle C to the opposite angle D: The figure ADBC is a parallelogram.

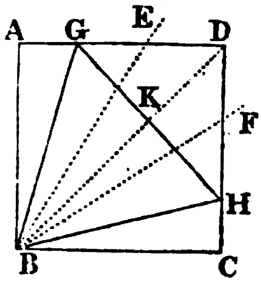


For, (E. 32. 1. Cor. 1.) the four angles of the figure ADBC are together equal to four right ∠∠; and, by the hypothesis, the four ∠∠ are the double of the two ∠∠, DAC, ACB; it is manifest, ∴, that the two ∠∠ DAC, ACB are together equal to two right ∠∠; ∴ (E. 28. 1.) AD is parallel to CB: And, in the same manner, AC may be shewn to be parallel to DB; ∴ the figure ADBC is a parallelogram.

Prop. XXXVII.

51. Problem. *In a given square to inscribe an equilateral triangle, having one of its angular points upon one of the angular points of the square, and its two remaining angular points one in each of two adjacent sides of the square.*

Let ABCD be a given square: It is required



to inscribe in it an equilateral triangle, having one of its angular points upon the angular point B of the square.

Trisect (S. 20. 1.) the right ∠ ABC, by $\overline{BE}$ and $\overline{BF}$; bisect (E. 9. 1.) the ∠∠ ABE, CBF by $\overline{BG}$ and $\overline{BH}$, meeting AD and DC in G and H, respectively; and join G, H: The △ GBH is equilateral.

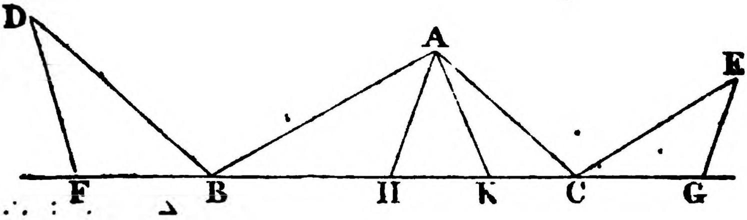
For, join B, D, and let BD meet GH in K: Then, it is manifest from the construction, that the ∠ ABG = ∠ CBH; also, (*hyp.* and E. 30. def. 1.) the ∠ A = ∠ C, and the side AB, of the △ ABG, is equal to the side CB, of the △ CBH; ∴ (E. 26. 1.) BG = BH; ∴ (E. 5. 1.) the ∠ BGH = ∠ BHG; also, (*constr.* and E. 8. 1.) the ∠ GBD = HBD; and BK is common to the two △△ BKG, BKH; ∴ (E. 26. 1.) the ∠ BKG = ∠ BKH; ∴ (E. 10. def. 1.) each of these ∠∠ is a right ∠; ∴ (E. 32. 1.) ∠ KGB + ∠ GBK = a right ∠ = ∠ GBH + 2 ∠ ABG = ∠ GBH + ∠ ABE (*constr.*); but, since (*constr.*) ∠ ABG + ∠ CBH = ∠ EBF, add to each of these equals the ∠∠ EBG, FBH, and the ∠ GBH = 2 ∠ ABE; ∴ the ∠ GBK = ∠ ABE; and it has been shewn that ∠ KGB + ∠ GBK = ∠ GBH + ∠ ABE; ∴ ∠ KGB = ∠ GBH; ∴ HG = GB = BH; *i. e.* the △ GBH is equilateral.

PROP. XXXVIII.

52. **THEOREM.** *If, at the extremities of the base of a given triangle, two straight lines be drawn, both above the base, and each of them equal to the adjacent side, and making with it an angle*

equal to the vertical angle of the triangle; then, if two straight lines, let fall from the extremities of the two so drawn, make, with the base produced, two angles that are equal each of them to the vertical angle, they shall cut off equal segments from the base produced.

From the extremities B, C, of the base BC of



the given △ ABC, let $\overline{BD}$ be drawn equal to the adjacent side AB, and $\overline{CE}$ equal to the adjacent side AC, making the ∠∠ ABD, ACE, each equal to the vertical ∠ BAC of the △, and let $\overline{DF}$ and $\overline{EG}$, drawn from D and E, make with BC produced the ∠∠ DFB, EGC, each also equal to the ∠ BAC: Then shall $\overline{FB}=\overline{GC}$.

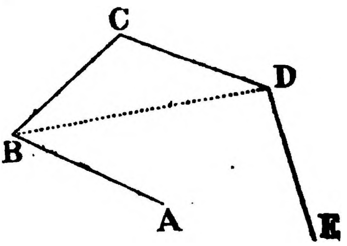
For, from the point A draw (S. 25. 1.) $\overline{AH}$ and $\overline{AK}$ making with BC the ∠∠ AHB, AKC each equal to the ∠ DFB, or BAC, or CGE: And, since (E. 13. 1.) ∠ ABH + ∠ ABD + ∠ DBF = two right ∠∠ = ∠ DBF + ∠ BFD + ∠ FDB (E. 32. 1.), and that (*constr.*) ∠ ABD = ∠ BFD, ∴ ∠ ABH = ∠ FDB; but, (*constr.*) ∠ AHB = ∠ DFB, and the side AB of the △ AHB is equal to the side DB of the △ DFB; ∴ (E. 26. 1.) $\overline{FB}=\overline{AH}$: And in the same manner $\overline{GC}$ may be shewn to be equal to $\overline{AK}$; but since (*constr.*) the $\angle AHB = \angle AKC$, ∴ (E. 13. 1.) the $\angle AHK = \angle AKH$; ∴ (E. 6. 1.) $AH = AK$; and FB was shewn to be equal to $\overline{AH}$, and $\overline{GC}$ to $\overline{AK}$; ∴ $\overline{FB} = \overline{GC}$.

53. Cor. If the vertical $\angle BAC$ be a right ∠, the two straight lines AH and AK coincide; and the segments FB, GC are equal each of them to the perpendicular drawn from A to the base BC: In this case, also, $\overline{DF} = \overline{BK}$, and $\overline{EG} = \overline{CK}$.

Prop. XXXIX.

54. Theorem. *If four straight lines cut each other, without including space, but so as to make three internal angles, towards the same parts, which together are less than four right angles, the two lines, which are not joined, shall meet, if produced far enough.*

Let the four straight lines AB, BC, CD, DE,



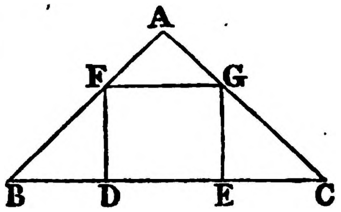
cut one another, without enclosing space, so that the ∠∠ ABC, BCD, CDE, are together less than four right ∠∠; Then shall BA and DE meet, if they are produced far enough.

For, join B, D: And (E. 32. 1.) $\angle DBC + \angle BCD + \angle CDB =$ two right ∠∠; if, ∴, these three ∠∠ be taken from the three given ∠∠, which (*hyp.*) are less than four right ∠∠, there will remain the two ∠∠ ABD, EDB, together less than two right ∠∠; ∴ (E. 12. axiom 1.) BA and DE will meet if they be continually produced.

PROP. XL.

55. PROBLEM. *To inscribe a square in a given right-angled isosceles triangle.*

Let ABC be the given isosceles △, right-angled



at A: It is required to inscribe a square in the △ ABC.

Trisect (S. 33. 1.) the hypotenuse BC, in the points D and E; from D and E draw (E. 11. 1.) $\overline{DF}$ and $\overline{EG}$ ⊥ to BC, meeting the sides AB and AC in F, and G, respectively; and join F, G: The inscribed figure FDEG is a square.

For, since the ∠ A is a right-angle, and that (*hyp.* and E. 5. 1.) $\angle B = \angle C$, ∴ (E. 32. 1.) ∠ B is half a right-angle; but (*constr.*) the ∠ D is a right ∠; ∴ the ∠ DFB is half a right ∠, and is, ∴, equal to the ∠ FBD; ∴ (E. 6. 1.) DF = BD; but (*constr.*) BD = DE; ∴ DF = DE; and, in the same manner, it may be shewn that EG = DE; ∴ DF = DE = EG.

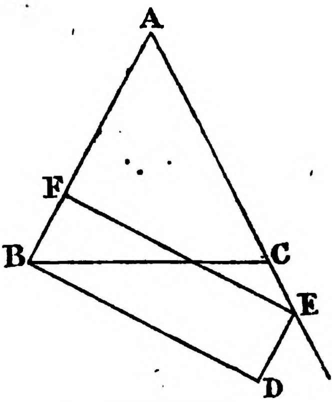
Again, since (*constr.*) the ∠∠ D and E are right ∠∠, ∴ (E. 28. 1.) DF is parallel to EG; and it has been shewn that DF = EG = DE; ∴ (E. 33. 1.) FG is equal and parallel to DE; ∴ (E. 29. 1.) the figure FDEG has all its ∠∠ right ∠∠; and it is equilateral; ∴ (E. 30. def. 1.) it is a square.

PROP. XLI.

56. PROBLEM. *To find a point, in either of the equal sides of a given isosceles triangle, from which, if a straight line be drawn, perpendicular to that side, so as to meet the other side produced, it shall be equal to the base of the triangle.*

Let ABC be the given isosceles △: It is required to find, in either of the two equal sides, as AB, a point from which if a perpendicular be drawn to AB and produced to meet AC, produced, it shall be equal to the base BC.

Draw (E. 11. 1.) from B, BD ⊥ to AB, and make (E. 3. 1.) BD = BC; from D draw (E. 31. 1.) DE parallel to AB, meeting AC produced in E;



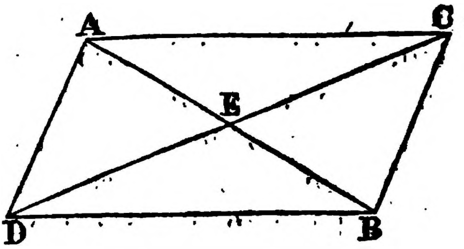
and from E, draw EF parallel to BD: F is the point which was to be found.

For (*constr.*) the figure FBDE is a ▱; ∴ (E. 34. 1.) FE = BD = BC (*constr.*); also, since (*constr.*) the ∠ FBD is a right ∠, the ∠ BFE is, also, (E. 29. 1.) a right ∠.

Prop. XLII.

47. Theorem. *The diameters of a parallelogram bisect each other.*

Let AB and CD be the diameters of the ▱



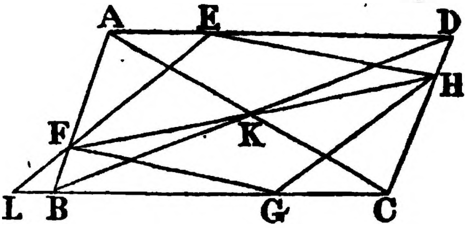
ADBC; AB and CD bisect one another in the point of their intersection E.

For since ADBC is a ▭, AD = CB (E. 34. 1.) and (E. 29. 1.) the ∠ EAD of the △ AED, = ∠ EBC, of the △ BEC, and the ∠ EDA = ∠ ECB; ∴ (E. 26. 1.) AE = EB, and DE = EC.

Prop. XLIII.

58. Theorem. *If in two opposite sides of a parallelogram two points be assumed, one in each of those sides, equidistant from two opposite angles of the figure, and if two other points be likewise assumed, in the two other opposite sides, equidistant from the same two angles, the figure, contained by the straight lines joining the four points so assumed, shall be a parallelogram.*

In the opposite sides AD, BC of the ▭ ABCD,



let the points E and G be taken equidistant from the opposite ∠s A and C; let also, the points F and H be taken, in the other two opposite sides, AB and DC, equidistant from A and C; and let E, F, and F, G, and G, H, and H, E, be joined: The figure EFGH is a parallelogram.

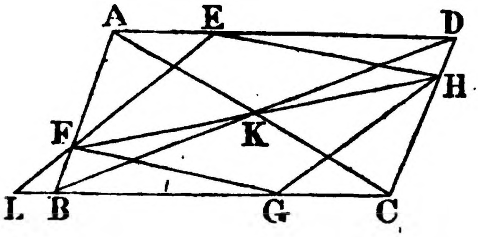
For since (*hyp.*) AE = CG, and AF = CH, and that (E. 34. 1.) the ∠ A = ∠ C, ∴ (E. 4. 1.) FE =

GH: Again since (E. 34. 1.) AB = DC, and AD = BC, and that (*hyp.*) of AD the part AE is equal to the part CG of BC, and of AB the part AF is equal to the part CH of DC, ∴ ED = GB, and DH = BF; also (E. 34. 1.) the ∠EDH = ∠FBG; ∴ (E. 4. 1.) EH = FG; and it has been proved that EF = HG; ∴ (S. 18. 1.) EFGH is a parallelogram.

PROP. XLIV.

59. THEOREM. *If any number of parallelograms be inscribed in a given parallelogram, the diameters of all the figures shall cut one another in the same point.*

Let ABCD be a given ▱, and let EFGH be



any ▱ whatever, inscribed in ABCD: The diameters of ABCD and of EFGH cut one another in the same point.

For draw AC a diameter of ABCD, and FH a diameter of EFGH; let AC and FH cut one another in K; and let CB, produced, meet EF, produced, in L: Then, since AE is parallel to BC, and EF parallel to HG, the ∠CGH = (E. 29. 1.)

∠GLE; and the ∠GLE = ∠LEA; ∴ the ∠CGH = ∠AEF; also (*hyp.* and E. 34. 1.) the ∠A = ∠C, and the side FE = the opposite side GH, of the ▱ EFG; ∴ (E. 26. 1.) CH = AF: Again, since the side AF of the △ AKF = the side CH of the △ CKH, and that (E. 29. 1.) the ∠∠ KAF, KFA are equal to the ∠∠ KCH, KHC, ∴ (E. 26. 1.) AK = KC, and FK = KH; *i. e.* K is the bisection of the diameters AC, FH; ∴ (S. 42. 1.) all the diameters cut one another in the point K.

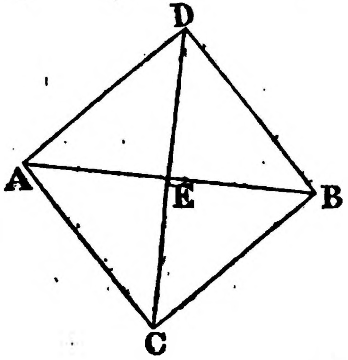
60. Cor. From the demonstration it is manifest, that the angle contained by any two given straight lines, is equal to the angle contained by two other straight lines, that are parallel to the two given straight lines, each to each.

Prop. XLV.

61. Theorem. *The diameters of an equilateral four-sided plane rectilineal figure bisect one another at right angles.*

Let AB and DC be the diameters of the equilateral four-sided figure ACBD, cutting one another in E: AB and DC bisect one another in E, at right angles.

For, since (*hyp.*) ACBD is equilateral, it is (S. 18. 1.) a ▱; and ∴ (S. 42. 1.) the diameters bisect one another in E: Again, because DE = CE, and EA is common to the two △△ AED,

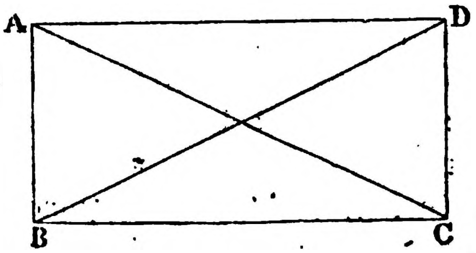


AEC, and that (*hyp.*) AD = AC, ∴ (E. 8. 1.) the ∠ AED = ∠ AEC; *i. e.* (E. 10. def. 1.) each of the ∠∠ AED, AEC is a right ∠; ∴ (E. 15. 1.) each of the ∠∠ DEB, CEB, is, also, a right ∠.

Prop. XLVI.

62. Theorem. *The diameters of a rectangle are equal to one another.*

Let AC, and BD be the diameters of the rectangle ABCD: Then AC = BD.

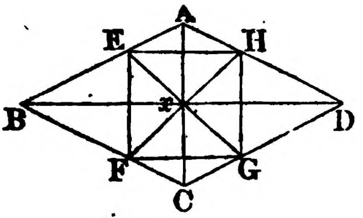


For, since (*hyp.*) the opposite ∠∠ of the figure are equal, each being a right ∠, ∴ (S. 26. 1.) the figure ABCD is a ▱; ∴ (E. 34. 1.) AD = BC; and AB is common to the two △ ABC, BAD; and the ∠ ABC = ∠ BAD; ∴ (E. 4. 1.) AC = BD.

PROP. XLVII.

63. **PROBLEM.** *To inscribe a square in a given equilateral four-sided figure.*

Let ABCD be the given equilateral four-sided figure: It is required to inscribe a square in ABCD.



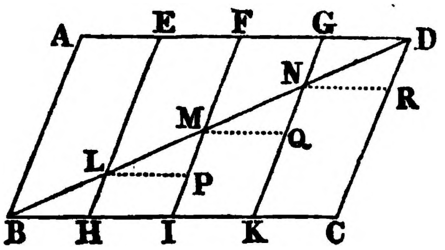
Join A, C, and B, D, and let $\overline{AC}$ and $\overline{BD}$ cut one another in X; bisect (E. 9. 1. and E. 15. 1.) the ∠∠ AXB and CXD, by the straight line EG, and the ∠∠ BHC, AXD, by the straight line FH; and join E, F, and F, G, and G, H, and H, E: The inscribed figure EFGH is a square.

For, since the figure ABCD is (*hyp.*) equilateral, AC and BD (S. 45. 1.) bisect one another at right ∠∠; ∴ (*constr.*) each of the ∠∠ EXA, AXH, HXD, DXG, GXC, CXF, FXB, BXE, is half a right ∠; ∴ the ∠∠ EXH, HXG, GXF, FXE are right ∠∠: Again, because ABCD is equilateral, it is (S. 18. 1.) a ▭; ∴ (E. 28. 1.) the ∠ BAC = ∠ ACD; but, because (*hyp.*) DC = DA, ∴ (E. 5. 1.) the ∠ ACD = ∠ DAC; ∴ the ∠ BAC = ∠ DAC; *i. e.* the ∠ EAX = ∠ HAX; and it has been shewn that the EXA = ∠ HXA; and AX is common to the two △ AEX, AHX; ∴ (E. 26. 1.) EX = HX; and, in the same manner it may be shewn that EX, HX, GX, and FX, are all equal: and the ∠∠ contained by those lines are equal, being right ∠∠; ∴ (E. 4. 1.) the figure EFGH is equilateral, and ∴ (S. 18. 1.) it is a ▭; and since the ∠ EXH is a right ∠, and that XE = XH, ∴ (E. 5. 1. and E. 32. 1.) each of the ∠∠ XEH, XHE is half a right ∠: In the same manner it may be shewn that each of the ∠∠ XHG, XGH, XGF, XFG, XFE, XEF is half a right ∠; ∴ the figure EFGH, which has been shewn to be equilateral, has all its ∠∠ right angles; ∴ (E. 30. def. 1.) it is a square.

PROP. XLVIII.

64. THEOREM. *If two opposite sides of a parallelogram be divided each into the same number of equal parts, the straight lines, joining the opposite points of division, shall also divide the diameter of the parallelogram into the same number of equal parts.*

Let the two opposite sides AD, BC, of the ▭ ABCD, of which BD is a diameter, be divided



into any number of equal parts, AE, EF, FG &c., BH, HI, IK &c.; and let E, H, and F, I, and G, K, &c. be joined: The diameter BD is divided by $\overline{EH}$, $\overline{FI}$, $\overline{GK}$, &c. into the same number of equal parts, BL, LM, MN, &c. as either of the opposite sides AD, or BC.

For, through L, M, N, &c. draw (E. 31. 1.) LP, MQ, NR, &c. each parallel to AD or BC: Then, since AE is equal and parallel to BH, EH (E. 33. 1.) is parallel to AB; and in the same manner it may be shewn, that FI, GK, &c., are parallel to one another; ∴ the figures LI, MK, NC, &c., are ▱; ∴ (E. 34. 1.) LP = HI; but (*hyp.*) HI = BH; ∴, BH = LP; and since LP is parallel to BC, and LH parallel to MI, and that $\overline{MLB}$ meets these parallels, ∴ (E. 29. 1.) the ∠ HBL, BLH, of the △ BLH, are equal to the ∠ PLM, LMP, of the △ LMP; and it has been proved that the side BH = LP; ∴ (E. 26. 1.) BL = LM: And, in the same manner it may be shewn, that LM = MN; MN = ND, and so on.

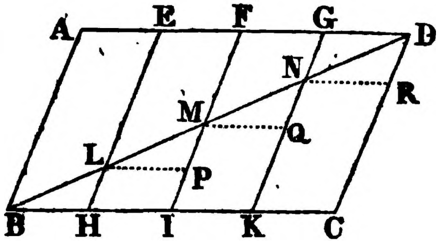
65. Cor. From the demonstration it is manifest that if the one of two given straight lines, or

a part of it, be divided into any number of equal parts; and from the points of division parallel straight lines be drawn cutting the other, the segments of that other given line, between these parallels, will be equal to one another.

PROP. XLIX.

66. PROBLEM. *To divide a given finite straight line into any given number of equal parts.*

Let BD be a given finite straight line: It is required to divide it into any given number of equal parts.



From B draw an indefinite straight line BC making any angle with DB; and from D draw (E. 31. 1.) DA, also indefinite, and parallel to BC; take any point H in BC; make (E. 3. 1.) HI, IK, DG, GF, FE, each equal to BH, so that the number of these equal straight lines in BC, and also in DA, may be less by one than the given number of parts, into which BD is to be divided; and join

E, H, and F, I and G, K: The straight lines EH, FI, and GK, will divide BD into the required number of equal parts.

For, in BC, and DA, take KC, and EA, each equal to BH, and join A, B and D, C: Then, since (*constr.*) AD is equal and parallel to BC, AB is also (E. 33. 1.) parallel to DC; ∴ ABCD is a ▭, of which BD is a diameter; ∴ (*constr.* and S. 48. 1.) BD is divided by EH, FI, and GK, into as many equal parts as BC, or AD, is divided into.

Otherwise,

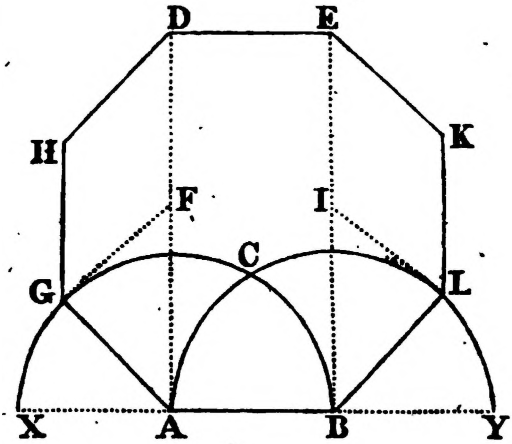
Draw BC, as before, and make the number of equal parts BH, HI, IK, KC, equal to the given number into which BD is to be divided; join C, D; and draw HL, IM, KN, each parallel to CD: Then will these parallels divide BD into the required number of equal parts.

For, if LP, MQ, NR be drawn each parallel to BC, it may be proved, (as in S. 48. 1.) that BL = LM = MN = ND.

PROP. L.

67. PROBLEM. *Upon a given finite straight line to describe an equilateral and equiangular octagon.*

Let AB be a given finite straight line: Upon AB, it is required to describe an equilateral and equiangular octagon.



From the points A and B draw (E. 11. 1.) AD and BE ⊥ to AB, and produce AB both ways to X and Y; bisect (E. 9. 1.) the ∠∠ DAX, EBY, by $\overline{AG}$, and $\overline{BL}$, and make $\overline{AG}$ and $\overline{BL}$ each equal to AB; from the points G and L, draw GF ⊥ to AG, and LI ⊥ to BL; also, draw (E. 31. 1.) GH parallel to AD, and make GH = AB or AG; in like manner, draw LK parallel to BE and make LK = AB; lastly, draw HD parallel to GF, meeting AD in D, and KE parallel to LI, meeting BE in E; and join D, E: The figure ABLKEDHG, described on AB, is an equilateral and equiangular octagon.

For, since (*constr.*) the side AG, of the △ AGF, is equal to the side BL, of the △ BLI, and ∠GAF = ∠LBI, and that the ∠∠ BGF, BLF are equal, being right ∠∠; ∴, (E. 26. 1.) AF = BI; also, since (*constr.*) HF and KI are ▱, FD = GH, and IE = LK; but (*constr.*) GH = LK; ∴

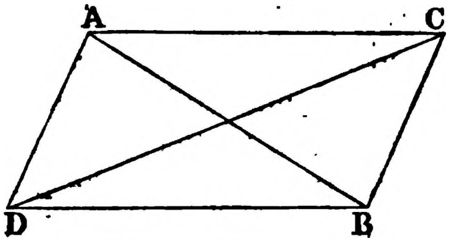
FD = IE; ∴ the whole AD = the whole BE; and (*constr.* and E. 28. 1.) AD is parallel to BE; ∴ (E. 33. 1.) DE is equal and parallel to AB; and ∴ (*constr.* and E. 34. 1.) the ∠∠ ADE, BED are right ∠∠: Again, since (*constr.*) the ∠∠ AGF, BLI, are right ∠∠, and that the ∠∠ GAF, IBL are each the half of a right ∠, ∴ (E. 32. 1.) the ∠∠ GFA, LIB, are each the half of a right ∠; ∴ AG = GF = BL = LI; and (E. 34. 1.) HD = GF, and KE = LI; whence it is manifest that the figure ABLKEDHG is equilateral.

Lastly, since HG is parallel to DA, and KL to EB, and FG and IL meet these parallels, ∴ (E. 29. 1.) the ∠ HGF = ∠ GFA, and ∴ ∠ HGF = the half of a right ∠; ∴ (E. 34. 1.) the ∠ HDF is the half of a right ∠; in the same manner, it may be shewn that each of the ∠∠ IEK, KLI, is the half of a right ∠; and it has been proved that the ∠∠ ADE, BED are right ∠∠; whence, and from the construction, it is manifest, that the figure ABLKEDHG, which has been shewn to be equilateral, is also equiangular.

PROP. LI.

68. THEOREM. *If either diameter of a parallelogram be equal to a side of the figure, the other diameter shall be greater than any side of the figure.*

Let the diameter AB, of the ▭ ACBD, be



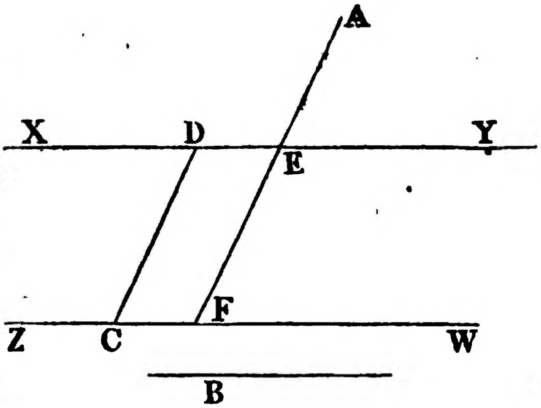
equal to the side AC: The other diameter CD shall be greater than either AC or AD.

For, because $AC = AB$, the $\angle ACB = \angle ABC$ (E. 5. 1.) and (*hyp.* and E. 29. 1.) the $\angle DAB = \angle ABC$; but the $\angle DAC > \angle DAB$; $\therefore$ the $\angle DAC > \angle ACB$; and the sides DA, AC, of the $\triangle$ DAC, are (E. 34. 1.) equal to the sides BC, CA, of the $\triangle$ BCA; $\therefore$ (E. 24. 1.) $CD > AB$; but (*hyp.*) $AB = AC$; $\therefore CD > AC$: And it has been shewn that the $\angle DAC > \angle ACB$; much more then is the $\angle DAC > \angle ACD$; $\therefore$ (E. 19. 1.) $DC > AD$.

PROP. LII.

69. PROBLEM. *From a given point to draw a straight line cutting two parallel straight lines, so that the part of it, intercepted between them, shall be equal to a given finite straight line, not less than the perpendicular distance of the two parallels.*

Let A be a given point; XY and ZW two given parallel straight lines, indefinite in length; and B



a given finite straight line, not less than the perpendicular distance of XY from ZW: It is required to draw through A, a straight line, cutting XY and ZW, so that the part of it, between the two parallels, shall be equal to B.

Take any point C in ZW; from C as a centre, at a distance equal to B, describe a circle, cutting in XY in D; join C, D; and through A draw (E. 34. 1.) $\overline{AF}$ parallel to DC, cutting XY and ZW in the points E and F: Then is $\overline{EF} = B$.

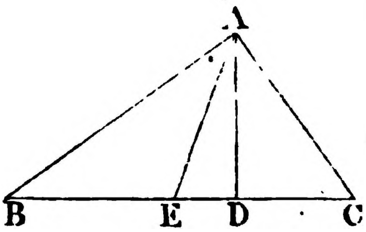
For, (*constr.*) the figure DCFE is a ▭; ∴ (E. 34. 1.) EF = DC; and (*constr.*) DC = B; ∴ EF = B.

PROP. LIII.

70. THEOREM. *If, from the summit of the right angle of a scalene right-angled triangle, two straight lines be drawn, one perpendicular to the*

hypotenuse, and the other bisecting it, they shall contain an angle equal to the difference of the two acute angles of the triangle.

Let the ∠ A, of the △ BAC be a right ∠; let



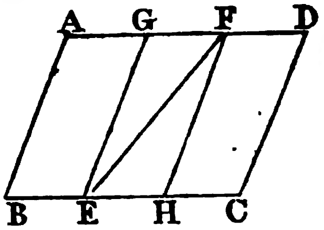
$\overline{AE}$ be drawn to the bisection E, of the hypotenuse BC, and let $\overline{AD}$ be drawn perpendicular to BC: The ∠ EAD = ∠ C − ∠ B.

For (S. 29. 1. and *hyp.*) EA = EB; ∴ the ∠ EAB = ∠ EBA: Again, since (*hyp.*) the two ∠∠ BAC, CBA, of the △ BAC, are equal to the two ∠∠ BDA, ABD, of the △ ADB, ∴ (S. 26. 1.) the ∠ BAD = ∠ ACB; but the ∠ EAD = ∠ BAD − ∠ BAE; ∴ the ∠ EAD = ∠ C − ∠ B.

PROP. LIV.

71. PROBLEM. *To bisect a parallelogram by a straight line drawn through a given point in one of its sides.*

Let ABCD be a ▱, and E a given point in one of its sides: It is required to bisect the ▱ ABCD, by a straight line drawn through E.



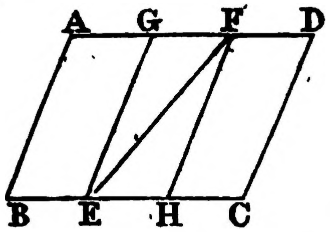
From AD, the side opposite to BC, cut off DF = BE (E. 3. 1.); and join E, F; $\overline{EF}$ bisects the ▱ ABCD.

For, through E and F draw (E. 31. 1.) EG and FH, each parallel to AB or DC; ∴ AE, GH, FC are ▱▱; and since EF is the diameter of the ▱ GH, ∴ (E. 34. 1.) the △ EGF = △ EHF; also, because BE = FD, and that AD is parallel to BC, ∴ (E. 36. 1.) the ▱ AE = ▱ FC; to these equals add the equal △△, EGF, EHF, and it is evident that the trapezium ABEF is equal to the trapezium FECD; *i. e.* $\overline{EF}$ bisects the ▱ ABCD.

PROP. LV.

72. THEOREM. *A trapezium, which has two of its sides parallel, is the half of a rectangle between the same parallels, and having its base equal to the aggregate of the two parallel sides of the trapezium.*

Let ABEF be a trapezium, having its side AF parallel to the opposite side BE; The trapezium ABEF is equal to the half of a rectangle between



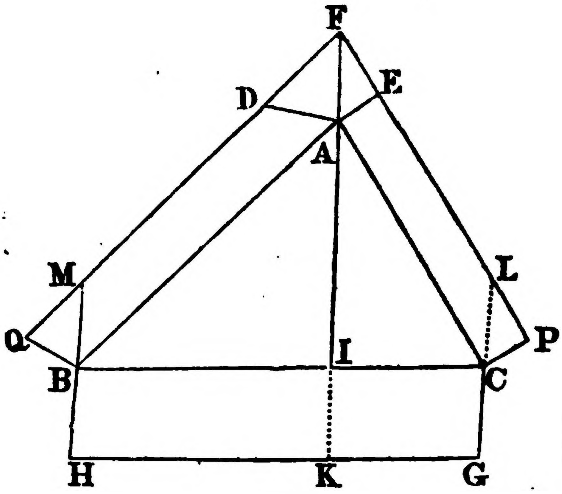
AF and BC, and having its base equal to $AF + BE$.

For, produce BE to C, and make $EC = AF$; through C draw (E. 31. 1.) CD parallel to BA, and let CD meet AF produced in D; ∴ the figure ABCD is a ▭; ∴ (E. 34. 1.) $AD = BC$; and (*constr.*) $AF = EC$; ∴ $FD = BE$: It is manifest, ∴, (from S. 54. 1.) that the trapezium ABEF is the half of the ▭ ABCD; but (E. 35. 1.) the ▭ ABCD = a rectangle upon the same base BC, and between the same two parallels; ∴ the trapezium ABEF = the half of a rectangle on the base BC, which (*constr.*) $= BE + AF$, and between the two parallels BE and AF.

PROP. LVI.

73. PROBLEM. *Any two parallelograms having been described on two sides of a given triangle, to apply, to the remaining side, a parallelogram, which shall be equal to their aggregate.*

Let the ▭▭ AQ and AP be on the two sides AB, AC, of the given △ ABC: It is required to apply



to the remaining side BC, a ▭ which shall be equal to the ▭ AP together with the ▭ AQ.

Produce QD and PE until they meet in F; join F, A; through C draw (E. 31. 1.) CG parallel to FA, and make, also, CG = FA; complete the ▭ BCGH: The ▭ BCGH = ▭ AQ + ▭ AP.

For, produce FA, so that it shall meet BC in I, and HG in K; produce, also, GC and HB, until they meet EP and DQ in L and M; ∴ the figures FACL, FABM are ◫; and, since (*constr.*) the ◫ FACL, CGKI, are upon equal bases FA, CG and between the same parallels, ∴ (E. 36. 1.) the ▭ FACL = ▭ CGKI; but (E. 35. 1.) the ▭ FACL = ▭ AP; ∴ the ▭ AP, = ▭ GI: And in the same manner, it may be proved that the ▭ AQ = ▭ IH; ∴ the whole ▭ BCGH = ▭ AP + ▭ AQ.*

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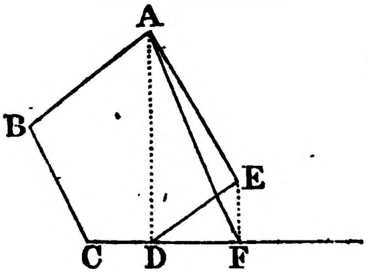
* If the parallelograms AP and AQ are squares, it is easy

PROP. LVII.

74. **PROBLEM.** *A plane rectilineal figure of any number of sides being given, to find an equal rectilineal figure, which shall have the number of its sides less, or greater, by one, than that of the given figure.*

First, let ABCDE be a given rectilineal figure:



It is required to find an equal rectilineal figure, having the number of its sides less by one, than the number of the sides of ABCDE.

Let A, E, D be any three consecutive ∠ of the given figure ABCDE; join A, D; through E draw (E. 31. 1.) EF parallel to AD and meeting CD, produced, in F; join A, F: The figure ABCF, which has the number of its sides less by one than ABCDE, is equal to ABCDE.

For, since the two △ AED, AEF, are upon

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to shew that the parallelogram BG will also be a square; and thus the forty-seventh proposition of the first Book of Euclid's Elements will have been demonstrated.

the same base AD and (*constr.*) are between the same parallels, AD, EF, ∴ (E. 37. 1.) the △ AFD = △ AED; to each of these equals add the figure ABCD; and the figure ABCF = the figure ABCDE.

Secondly, let ABCF be a given rectilineal figure; and let it be required to find an equal rectilineal figure, having more sides by one, than ABCF.

Take any point, D, in any of the sides, as CF, of ABCF, and join B, D, or A, D; A and B being the ∠∠ which are next to the ∠∠ F and C, at the extremities of CF; then, A, D having been joined, through F draw (E. 31. 1.) FE parallel to DA; and since the ∠ ADC is greater (E. 16. 1.) than the ∠ ADF, and equal (E. 29. 1.) to the ∠ EFD, ∴ FE falls without the given figure: In FE take any point E, and join E, A, and E, D: The figure ABCDE has more sides, by one, than the given figure ABCF; and it may be shewn, as in the preceding case, to be equal to ABCF.

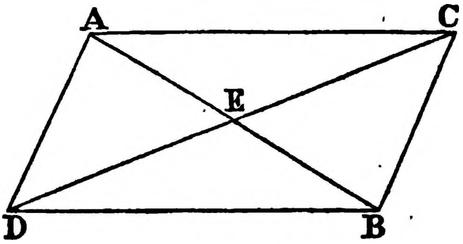
75. Cor. Hence, first, a triangle may be found which shall be equal to any given rectilineal figure: For the number of sides of the given figure being thus diminished, by one, at each step, they will at length be reduced to three, and the triangle which they contain, will be equal to the given figure.

Secondly, it is manifest, that, by the latter part of the preceding problem, a polygon, of any given number of sides, may be found, which shall be equal to a given triangle.

Prop. LVIII.

76. **Theorem.** *The diameters of any parallelogram divide it into four equal triangles.*

Let ADBC be a ▱, of which the diameters AB, CD cut one another in E: The four △ AED, DEB, BEC, CEA are equal to one another.



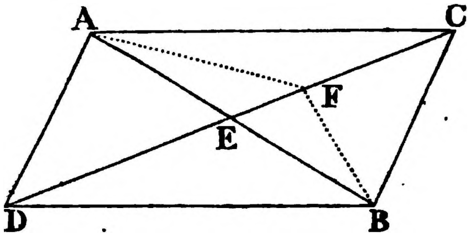
For (*hyp.* and E. 34. 1.) the side AC of the △ AEC, is equal to the side DB, of the △ DEB; also (S. 42. 1.) AE = EB, and DE = EC; (E. 8. 1. and E. 4. 1.) the △ AEC = △ DEB. In the same manner, it may be shewn that the △ AED = △ CEB: And since, the two △ AED, AEC, stand upon equal bases DE and EC, ∴ (E. 38. 1.) the △ AED = △ AEC. It is manifest, ∴, that the four △ AED, AEC, CEB, BED are equal to one another.

Prop. LIX.

77. **Problem.** *If two triangles have the two adjacent sides of a parallelogram for their bases, and*

have their common vertex situated in the diameter, or in the diameter produced, they shall be equal to one another.

Let the two △ AFC, BFC, have the two adjacent sides AC, BC, of the ▭ ADBC, for their bases, and also have their common vertex situated at any point F, in the diameter DC, or in DC, produced: The △ AFC = △ BFC.



First, let the point F be in the diameter DC: Join AB; and let AB cut DC in E.

Then, since (S. 42. 1.) AE = EB, ∴ (E. 38. 1.) the △ AEC = △ BEC, and the △ AEF = △ BEF; ∴ the △ AFC, BFC, which are the differences of these equals, are equal to one another.

And the proposition may, in the same manner, be shewn to be true, when the common vertex of the two △, which have AC and BC for their bases, is in DC produced.

Prop. LX.

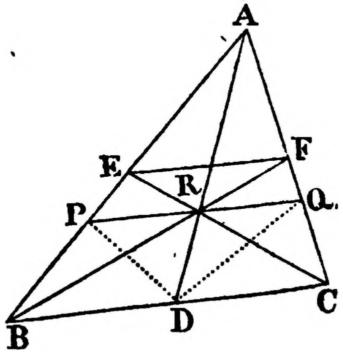
78. Theorem. *Of all triangles, which are between the same parallels, that which stands on the greatest base is the greatest.*

For it is manifest, that the Δ which has the greater base will exceed the Δ which is formed by joining its vertex and the extremity of a segment of its base made equal to the base of the other Δ : But the Δ so formed is equal (E. 38. 1.) to the other given Δ ; ∴ the Δ which has the greater base is greater than that other triangle.

PROP. LXI.

79. THEOREM. *The straight line, joining the vertex and the bisection of the base of any triangle, bisects every other straight line that is parallel to the base and is terminated by the two remaining sides of the triangle.*

Let $\overline{PQ}$ be any straight line, either within or without the Δ ABC, parallel to the base BC, and let $\overline{AD}$, joining the vertex A and the bisection D of $\overline{BC}$, cut $\overline{PQ}$ in R : $\overline{PQ}$ is bisected by $\overline{AD}$ in R.



First, let $\overline{PQ}$ be within the △ ABC; and if $\overline{PR}$ be not equal to $\overline{RQ}$, one of them is the greater: Let $\overline{PR} > \overline{RQ}$; and join D, P, and D, Q.

Then since (*hyp.*) the base BD, of the △ BAD, is equal to the base DC, of the △ CAD, ∴ (E. 38. 1.) the △ BAD = △ CAD; also, because $\overline{BD} = \overline{DC}$, and that (*hyp.*) PQ is parallel to BC, ∴ (E. 38. 1.) the △ BPD = △CQD; if, ∴, the two latter equal △s be taken from the equal △s BAD, CAD, there remains the △ APD = △ AQD: But, since PR > RQ, the △ APR > △ AQR, and the △ DPR > △ DQR; ∴, the whole △ APD > △ AQD; but it has been shewn that the △ APD = △ AQD; and it is, also, greater; which is absurd: ∴, neither of the two lines PR, RQ, can be greater than the other; ∴, PR = RQ.

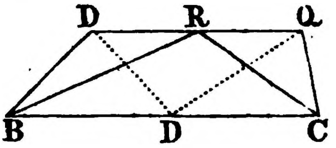
In a similar manner the proposition may be proved, when PQ is without the △ ABC.

80. Cor. Hence, it is easily shewn, *ex absurdo*, that the straight line joining the bisections of any two straight lines, that are parallel to the base, and terminated by the sides of a △, passes through the vertex of the △.

PROP. LXII.

81. THEOREM. *If two opposite sides of a trapezium be parallel to one another, the straight line, joining their bisections, bisects the trapezium.*

For, let PBCQ be a trapezium having the side



PQ parallel to BC, and let $\overline{RD}$ join the bisections, R and D, of the opposite sides PQ and BC: RD bisects the trapezium PBCQ.

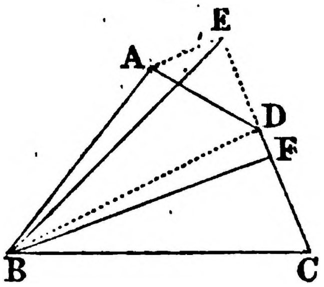
For, join P, D, and Q, D: Then since (*hyp.*) PQ is parallel to BC, and that the base BD of the △ BPD, is equal to the base DC, of the △ DQC, ∴ (E. 38. 1.) the △ BPD = △DQC; and, in the same manner, it may be shewn that the △ PDR = △ DRQ; ∴, △ BPD + △ PDR = △ DQC + △ DRQ; *i. e.* the figure BPRD = CQRD; ∴ RD bisects the trapezium PBCQ.

Prop. LXIII.

82. Problem. *To bisect a given trapezium by a straight line drawn from any of its angles.*

Let ABCD be a trapezium: It is required to draw a straight line from any of the ∠, as B, which shall bisect the trapezium ABCD.

Join B, D; through A draw (E. 31. 1.) AE parallel to BD, and let CD, produced, meet AE in E; bisect (E. 10. 1.) $\overline{EC}$ in F; and join B, F; $\overline{BF}$ bisects the trapezium ABCD.



For join B, E; and since the two △s BAD, BED are on the same base BD, and between the same parallels, ∴ (E. 37. 1.) the △ BAD = △ BED; to each of these equals add the △ BDF; ∴ △ BAD + △ BDF = △ BED + △ BDF; *i. e.* the trapezium BADF = △ BEF; but since (*constr.*) EF = FC, ∴ (E. 38. 1.) the △ BEF = △BFC; ∴ the trapezium BADF = △ BFC; *i.e.* BF bisects the given trapezium ABCD.

Prop. LXIV.

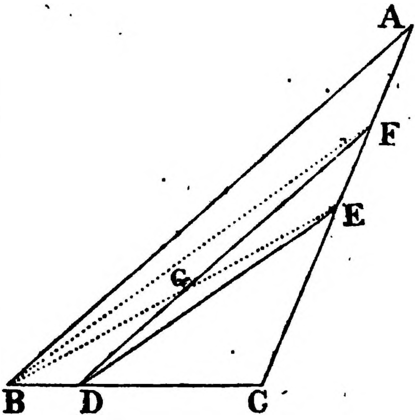
83. Problem. *To bisect a given triangle by a straight line drawn through a given point in any one of its sides.*

Let ABC be the given △, and let D be a given point in one of its sides BC: It is required to draw through D a straight line which shall bisect the triangle.

Bisect (E. 10. 1.) AC in E; join D, E; through

B draw (E. 31. 1.). BF parallel to DE, meeting AC in F; join D, F: DF bisects the △ ABC.

For join B, E and let BE cut DF in G: Then since the △s DFE, EBD are upon the same base



DE and (*constr.*) between the same parallels, ∴ (E. 37. 1.) the △ DFE = △ EBD; take away the common part DGE, and there remains the △ BGD = △ EGF; to each of these equals add the trapezium ABGF, and it is manifest that the trapezium ABDF = △ ABE; but since (*constr.*) AE = EC, ∴ (E. 38. 1.) the △ ABE = △ EBC; ∴ the trapezium ABDF is equal to the half of the given △ ABC; *i. e.* DF bisects the △ ABC.

Prop. LXV.

84. Problem. *Equal triangles, which have their bases in the same straight line and which are*

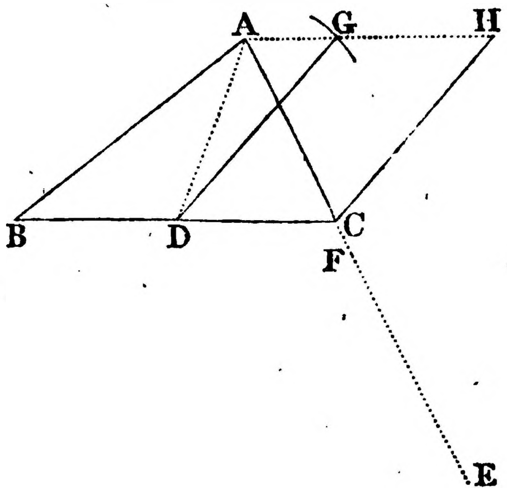
between the same parallels, stand upon equal bases.

For if not, let one of the bases be greater than the other; ∴ (S. 60. 1.) the △, of which it is the base, is greater than the other, which is contrary to the supposition : ∴, neither of the bases can be greater than the other; *i. e.* the bases are equal to one another.

PROP. LXVI.

85. PROBLEM. *To describe a parallelogram, the surface and perimeter of which shall be respectively equal to the surface and perimeter of a given triangle.*

Let ABC be the given △ : It is required to



describe a ▭, which shall be equal to the Δ ABC, and which shall, also, have its perimeter equal to the perimeter of ABC.

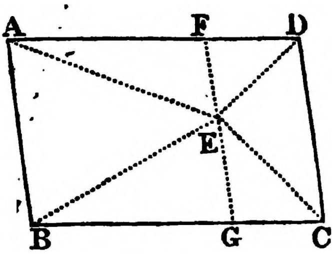
Bisect (E. 10. 1.) BC in D; produce AC to E, and make CE = AB; bisect AE in F; through A draw (E. 31. 1.) AH parallel to BC; from D as a centre, at a distance equal to AF describe a circle, cutting AH in G; join D, G, and through C draw CH parallel to DG: Then is the ▭ DCHG = Δ ABC, and the perimeter of DCHG is, also, equal to the perimeter of ABC.

For join A, D; and since BD = DC, ∴ (E. 38. 1.) the Δ ABD = Δ ACD, so that the whole Δ ABC is the double of the Δ ADC: Again, since the ▭ DCHG and the Δ ADC are on the same base DC, and between the same parallels, ∴ (E. 41. 1.) the ▭ DCHG is the double of the Δ ADC; as is, also, the Δ ABC: ∴ the ▭ DCHG = Δ ABC: And because (*constr.*) DG is equal to the half of BA + AC, and that (E. 34. 1.) CH = DG, ∴ DG + CH = BA + AC; also (E. 34. 1.) GH = DC = DB; ∴ DC + GH = BD + DC = BC; ∴ DG + GH + HC + CD = BA + AC + CB.

Prop. LXVII.

86. Theorem. *The two triangles formed by drawing straight lines, from any point within a parallelogram, to the extremities of either pair of opposite sides, are, together, half of the parallelogram.*

Let E be any point in the ▭ ABCD, and let



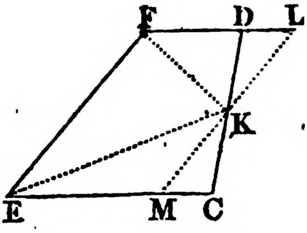
E, A, and E, B, and E, C and E, D be joined: The two △s AEB, DEC are, together, half of the ▭ ABCD.

For, through E draw (E: 31. 1.) $\overline{FEG}$ parallel to $\overline{AB}$ or $\overline{DC}$: and since AG, and GD are ▭, ∴ (E. 41. 1.) the △ AEB is the half of the ▭ AG, and the △ DEC is the half of the ▭ GD; ∴ the △ AEB + △ DEC is the half of the ▭ AG + the half of the ▭ GD, or the half of the whole ▭ ABCD.

Prop. LXVIII.

87. Theorem. *If two sides of a trapezium be parallel, the triangle contained by either of the other sides, and the two straight lines drawn from its extremities to the bisection of the opposite side, is the half of the trapezium.*

Let the two sides FD, EC, of the trapezium FECD be parallel; let K be the bisection of



either of the two remaining sides, as DC; and let K, E and K, F be joined:. The △ FKE is the half of the trapezium FECD.

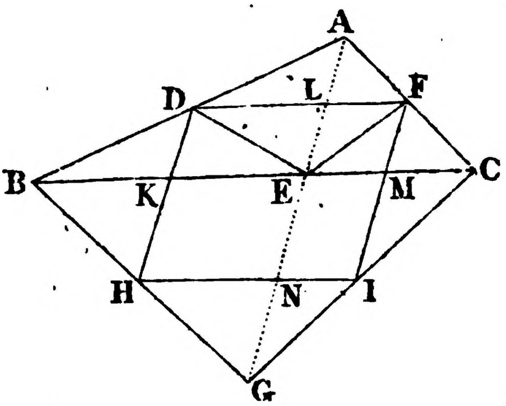
For, through K draw (E. 31. 1.) LKM parallel to FE, and let LKM meet BC in M and FD, produced, in L. And since FL (*constr.*) is parallel to EC, and LM meets them, ∴ the ∠ DLK = ∠ KMC; also (E. 15. 1.) the ∠ DKL = ∠ CKM, and (*hyp.*) the side DK of the △ DKL, = the side CK of the △ CKM; ∴ (E. 26. 1. and E. 4. 1.) the △ DKL = △ CKM; but if to the rectilineal figure FEMKD there be added the △ CKM, there results the trapezium FECD; and if to the same figure there be added the △ DKL, there results the ▭ FEML; ∴ these results are equal; but (E. 41. 1.) the △ FKE is the half of the ▭ FEML; ∴, the △ FKE is the half, also, of the trapezium FECD.

Prop. LXIX.

88. Theorem. *The triangle contained by the straight lines joining the points of the bisection of the three sides of a given triangle, is one-fourth*

part of the given triangle, and is equiangular with it.

Let D, E, F, be the bisections of the sides AB,



BC, CA, respectively, of the given △ ABC; and let D, E, and E, F, and F, D, be joined: The △ DEF is one fourth part of the △ ABC, and is equiangular with it.

For, join A, E; and, since (*hyp.*) BE = EC, CF = FA, and AD = DB, ∴ (E. 38. 1.) △ AEB = △AEC; and the △ AEB is the double of the △ BDE, and the △ AEC is the double of the △ CFE; ∴ the △ BDE = △ CFE; *i.e.* each of them is a fourth part of the △ ABC; also they are upon equal bases BE and EC; ∴ (E. 40. 1.) DF is parallel to BC; and, in the same manner, it may be shewn that DE is parallel to AC, and FE parallel to AB; ∴, the figures FCED, DBEF, are ▭; ∴ (E. 34. 1.) the △ DEF = △ DBE, which has been proved to be a fourth part of the

△ ABC; also, the ∠ DFE, of the ▱ BF, = opposite ∠ B, and the ∠ FDE, of the ▱ DC, = opposite ∠ C; ∴ (E. 32. 1.) the ∠ DEF, of the △ DFE = the ∠ BAC, of the △ ABC; and the two △s ABC, DEF, are ∴ equiangular.

89. Cor. 1. The straight line joining the bisections of any two sides of a △, is parallel to the remaining side.

90. Cor. 2. If the four sides of any given quadrilateral rectilineal figure be bisected, the figure contained by the straight lines joining the several points of the bisection, shall be a parallelogram, which is the half of the given figure; also the four sides of this parallelogram shall be, together, equal to the two diagonals of the given figure.

Let DH, HI, IF, FD be the straight lines joining the several bisections D, H, I, and F, of the sides AB, BG, GC, and CA, of the quadrilateral figure ABGC: The figure DHIF is a ▱; it is the half of the given figure ABGC; and its four sides are, together, equal to the two diagonals AG, BC, of the figure ABGC.

First, since, D, H, F, I, are the bisections of the sides of the △s ABG, GCA, BAC, CGB, ∴ (S. 69. 1. *cor.*) DH and FI are parallel to AG, and DF and HI are parallel to BC; ∴ (E. 30. 1.) DHIF is a ▱: And, because DF is parallel to BC, and AB meets them, ∴ (E. 29. 1.) the ∠ ADL = ∠ DBK; again, because DH is parallel to AG, and AB meets them, the ∠ DAL = ∠

BDK; and (*hyp.*) the side AD, of the △ ADL, = the side DB of the △ DBK; ∴ (E. 26. 1.) DL = BK, LA = KD, and (E. 4. 1.) the △ ADL = △ DBK; but DKEL being a ▭, DL = KE, and KD = EL (E. 34. 1.); ∴ BK = KE, and EL = LA: If, ∴, D, E be joined, the △ DLE = △ DLA (E. 38. 1.) and the △ DKE = △ DKB; so that the ▭ KL = the half of the △ AEB, DK + FM = AE, and DL + HN = BE. In the same manner it may be proved, that the ▭ LM = the half of the △ AEC, that the ▭ MN = the half of the △ CEG, that the ▭ NK = the half of the △ BEG, that LF + NI = EC, and that MI + KH = EG: ∴, the ▭ DHIF is the half of the given figure ABGC, and its four sides are, together, equal to the two diagonals AG, and BC.

91. Cor. 3. It is manifest that the straight lines which join the opposite points of bisection of the sides of any trapezium, bisect each other.

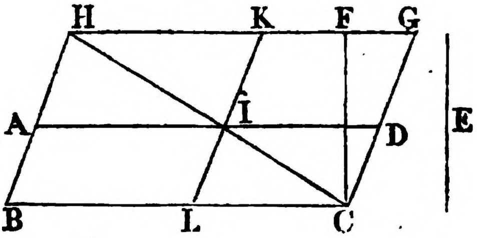
For, if D, I, and F, H, be the bisections of opposite sides of the given quadrilateral figure ABGC, it is manifest, from the preceding corollary, that the straight lines DI, FH which join them, will be the diameters of the ▭ DHIF; and ∴ (S. 42. 1.) they bisect one another.

Prop. LXX.

92. Problem. *To describe a parallelogram, which*

shall be of a given altitude, and equiangular with, and also equal to, a given parallelogram.

Let ABCD be a given ▭, and E a given straight line: It is required to describe a ▭ which shall be equal to the ▭ ABCD, and also equiangular with it; and which shall have its altitude equal to the given line E.



From the point C draw (E. 11. 1.) CF ⊥ to BC, and make CF=E; through F draw (E. 31. 1.) HG parallel to BC; produce BA and CD to meet HG, in H and G; join H, C, and let $\overline{HC}$ cut AD in I; through I draw (E. 31. 1.) KIL parallel to HB or GC: The ▭ KLCG, which (*constr.* and E. 29. and 34. 1.) is equiangular with the ▭ ABCD, and has its altitude equal to E, is also equal to the ▭ ABCD.

For, since BI and IG are compliments about the diameter HC of the ▭ HBCG, they are (E. 43. 1.) equal to one another; to each of these equals add the ▭LD; and it is plain that the ▭ KLCG=▭ ABCD.

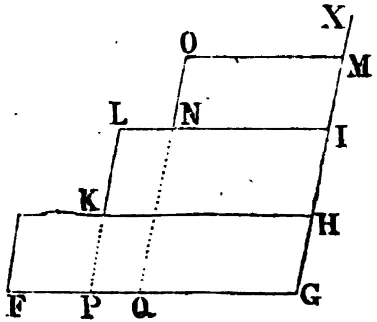
93. Cor. Hence, a rectangle may very readily be found, which shall be equal to a given square,

and shall have one of its sides equal to a given straight line.

Prop. LXXI.

94. Theorem. *If there be any number of rectilineal figures, of which the first is greater than the second, the second than the third, and so on, the first of them shall be equal to the last together with the aggregate of all the differences of the figures.*

First let there be three such given rectilineal figures. Make (E. 45. 1.) the ▭ FH equal to the greatest of the given figures, having its ∠ FGH of any given magnitude; produce GH to X; from HX cut off (E. 3. 1.) HI=GH; find (S. 57. 1. *cor.*) a △ equal to the next greatest of the given figures, and apply (E. 44. 1.) to HI a ▭ equal to that △, having its ∠ IHK = ∠ HGF: Again, from IX cut off IM=GH or HI, and, in like manner, to IM apply a ▭ IO, equal to the least of the given figures, and having its ∠ MIN= ∠ HGF.



Produce LK and ON to meet FG in P and Q; and let OQ meet KH in R.

Then, (E. 36. 1. E. 34. 1. and *constr.*) the ▭
FH = ▭ QH + ▭ PR + ▭ FK
i. e. the ▭ FH = ▭ NM + ▭ PR + ▭ FK.

But the ▭ PR is the difference of the ▭ PH and QH or (E. 36. 1.) of KI and NM; and the ▭ FK is the difference of the ▭ FH and PH, or of FH and KI: Whence it is manifest that the proposition is true, when three rectilineal figures are taken: And it may, in the same manner, be proved to be true, when more than three are taken.

Prop. LXXII.

95. Problem. *To find a rectangle, which shall have one of its sides equal to a given finite straight line, and which shall be equal to the excess of the greater of two given rectilineal figures above the less.*

To the given finite straight line, and on the same side of it, apply (E. 45. 1. *cor.*) two rectangles, the one equal to the greater and the other to the less, of the given rectilineal figures: And it is manifest that the rectangle which is the difference of the two rectangles so described, will have one of its sides equal to the given straight line, and will be equal to the excess of the greater of the two given figures above the less.

Prop. LXXIII.

96. Theorem. *If two right-angled triangles have two sides of the one equal to two sides of the other, each to each, the triangles shall be equal, and similar to each other.*

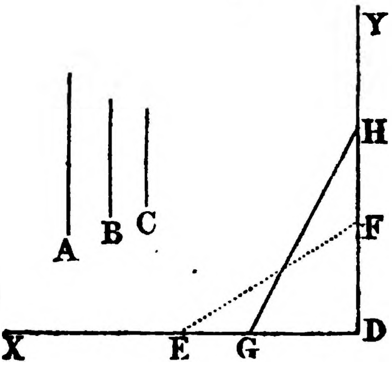
If the two sides about the right-angle of the one Δ, be equal to the two sides about the right-angle of the other, each to each, it follows, (from E. 4. 1.) that the Δs are equal and similar.

But, let now, the hypotenuses of the right ∠s, in the two Δs, be equal, and also let one other side, of the one Δ, be equal to another side of the other; ∴ (E. 47. 1.) the squares of the two remaining sides of the one, will be equal to the squares, taken together, of the two remaining sides of the other Δ; from these equals take away the equal squares of the two other sides, which, by the hypothesis, are equal, and there remains the square of the third side, of the one, equal to the square of the third side, of the other Δ; ∴ the third side of the one is equal to the third side of the other; ∴ (E. 4. 1.) the two Δs are equiangular, and are, also, equal to one another.

PROP. LXXIV.

97. PROBLEM. *To find a square which shall be equal to any number of given squares.*

First, let there be three given square, and let their sides be equal to the three straight lines A, B and C.



Take any straight line DX, indefinite towards X; from D draw (E. 11. 1.) $\overline{DY} \perp$ to $\overline{DX}$, and produce DY indefinitely towards Y: From DX cut off (E. 3. 1.) DE $=$ A, and from DY cut off DF $=$ B; and join E, F: Again, from DY cut off DH $=$ EF, and from DX cut off DG $=$ C, and join G, H: The squares described (E. 46. 1.) upon GH shall be equal to the three given squares to the sides of which A, B and C are respectively equal.

For (E. 47. 1. and *constr.*) $\overline{EF}^2 = \overline{ED}^2 + \overline{DF}^2$;
i. e. (*constr.*) $\overline{DH}^2 = A^2 + B^2$;

∴ (*constr.*) $\overline{DH}^2+\overline{DG}^2=A^2+B^2+C^2$

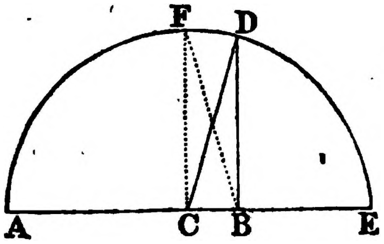
i. e. (E. 47. 1.) $\overline{GH}^2=A^2+B^2+C^2$.

And in the same manner, it is evident, a square may be found, which shall be equal to the aggregate of any number of given squares.

PROP. LXXV.

98. **PROBLEM.** *Two unequal squares being given, to find a third square, which shall be equal to the excess of the greater of them above the less.*

Let AC and CB placed in the same straight line, be the sides of the two given squares, of which the square of AC is the greater: From the centre C, at the distance CA, describe the circle ADE, meeting AB, produced, in E; from B draw (E. 11. 1.) $\overline{BD}$ ⊥ to AB, and let BD meet the circumference in D: The square of BD is equal to the excess of the square of AC above the square of BC.



For join D, C: And since (*constr.*) the ∠ B is a right ∠, ∴ (E. 47. 1.) $\overline{CD}^2=\overline{CB}^2+\overline{BD}^2$

i. e. (E. def. 15. 1.) $\overline{AC}^2=\overline{CB}^2+\overline{BD}^2$

Whence it is manifest, that the square of BD is equal to the excess of the square of AC above the square of CB.

PROP. LXXVI.

99. THEOREM. *If the side of a square be equal to the diameter of another square, the former square shall be the double of the latter.*

For (E. def. 30. 1. and E. 47. 1.) the square of the diameter of a square is equal to the squares of its two sides; *i. e.* to the double of the square itself: ∴ the square of any straight line which is equal to the diameter of a square, is the double of that square.

PROP. LXXVII.

100. THEOREM. *In any right-angled triangle, the square which is described on the side subtending the right angle, as a diameter, is equal to the squares described upon the other two sides, as diameters.*

For, (S. 76. 1.) the squares described on the hypotenuse, and on the two sides of a △ as diameters, are, respectively, the halves of the squares of those lines: But since (*hyp.*) the △ is right-angled, ∴ (E. 47. 1.) the square of the hypotenuse is equal to the squares of the two sides; ∴ the square described on the hypotenuse as a diameter, is equal to the squares described on the other two sides as diameters.

A

SUPPLEMENT

TO THE

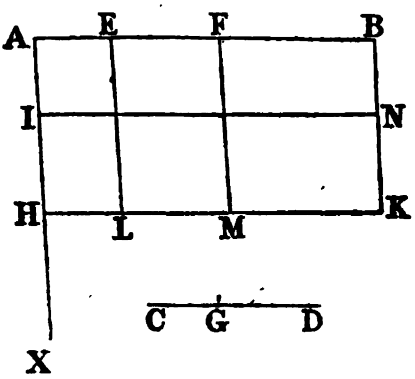
ELEMENTS OF EUCLID.

BOOK II.

Prop. I.

1. **Theorem.** *If two given straight lines be divided, each into any number of parts, the rectangle contained by the two straight lines, is equal to the rectangles contained by the several parts of the one and the several parts of the other.*

Let the given straight line AB be divided into



any parts in the points E, F, and let the given straight line CD be divided first into two parts in the point G: The rectangle contained by $\overline{AB}$ and $\overline{CD}$ is equal to rectangles contained by AE and CG, by EF and CG, by FB and CG, by AE and GD, by EF and GD, and by FB and GD, taken together.

From the point A draw (E. 11. 1.) $\overline{AX} \perp$ to AB; from AX cut off (E. 3. 1.) AI = CG, and from IX cut off IH = GD, so that AH = CD; through I and H draw (E. 31. 1.) $\overline{IN}$ and $\overline{HK}$ parallel to AB, and through B, F, E, draw $\overline{BK}$, $\overline{FM}$, $\overline{EL}$, parallel to AH: Then (E. 1. 2.) the rectangle AN is equal to the rectangles contained by AE and CG, by EF and CG, and by FB and CG; also the rectangle IK is equal to the rectangles contained by HL and GD, by LM and GD, and by MK and GD; but (E. 34. 1.) HL = AE; LM = EF; and MK = FB; ∴ the rectangle IK is equal to the rectangles contained by AE and GD, by EF and GD, and by FB and GD; but the two rectangles AN and IK make up the rectangle AK, which is contained by AB and AH or CD; ∴ the rectangle contained by AB and CD is equal to the rectangles contained by AE and CG, by EF and CG, by FB and CG, by AE and GD, by EF and GD, and by FB and GD, taken together.

And, in the same manner, the proposition may be proved to be true, when the given straight line CD is divided into more than two parts.

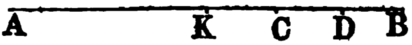
2. Cor. If the parts EF, FB, &c., into which

$\overline{AB}$ is divided, and the parts CG, GD, &c., into which $\overline{CD}$ is divided, be each of them equal to AE, it is manifest that the rectangle contained by AB and CD is equal to the square of AE taken as often as is indicated by the product of the number of equal parts in $\overline{AB}$, multiplied by the number of equal parts in CD.

PROP. II.

3. THEOREM. *If a straight line be divided into two unequal parts, in two different points, the rectangle contained by the two parts, which are the greatest and the least, is less than the rectangle contained by the other two parts; the squares of the two former parts, together, are greater than the squares of the two latter, taken together; and the difference between the squares of the former and the squares of the latter, is the double of the difference between the two rectangles.*

Let the given straight line AB be divided into



two unequal parts, in the point C, and also in the point D: Then $\overline{AD}\times\overline{DB} < \overline{AC}\times\overline{CB}$; but $\overline{AD}^2+\overline{DB}^2 > \overline{AC}^2+\overline{BC}^2$; and the excess of $\overline{AD}^2+\overline{DB}^2$ above $\overline{AC}^2+\overline{CB}^2$ is the double of the excess of $\overline{AC}\times\overline{CB}$ above $\overline{AD}\times\overline{DB}$.

For, bisect (E. 10. 1.) AB in K: Therefore,

$$\left.\begin{array}{r}\overline{AC}\times\overline{CB}+\overline{CK}^2=\overline{AK}^2\\ \text{and }\overline{AD}\times\overline{DB}+\overline{DK}^2=\overline{AK}^2\end{array}\right\}\text{(E. 5. 2.)}$$

But $CK^2 < DK^2$; $\therefore \overline{AD}\times\overline{DB} < AC\times CB$.

Again, because

$$\left.\begin{array}{l} AD^2+DB^2+2\overline{AD}\times\overline{DB}=AB^2 \\ AC^2+CB^2+2\overline{AC}\times\overline{CB}=AB^2 \end{array}\right\} \text{(E. 4. 2.)}$$

and that, as hath been shewn $\overline{AD}\times\overline{DB} < \overline{AC}\times\overline{CB}$,

$$\therefore AD^2+DB^2 > AC^2+CB^2.$$

Lastly, since

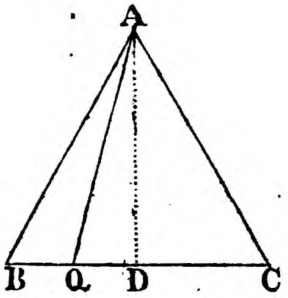
$$\overline{AD}^2+\overline{DB}^2+2\overline{AD}\times\overline{DB}$$
$$=\overline{AC}^2+\overline{CB}^2+2\overline{AC}\times\overline{CB},$$

it is manifest, if from these equals there be taken $AC^2+CB^2+>2\overline{AD}\times\overline{DB}$, that the excess of AD^2+DB^2 above AC^2+CB^2 is the double of the excess of $\overline{AC}\times\overline{CB}$ above $\overline{AD}\times\overline{DB}$.

PROP. III.

4. THEOREM. *In any isosceles triangle, if a straight line be drawn from the vertex to any point in the base, the square upon this line, together with the rectangle contained by the segments of the base, is equal to the square upon either of the equal sides.*

Let ABC be an isosceles △, and let $\overline{AQ}$, be



drawn from its vertex A, to any point Q, in BC its base: $\overline{AQ}^2+\overline{BQ}\times\overline{QC}=\overline{AB}$.

For bisect (E. 10. 1.) BC in D, and join A, D.

$\therefore \overline{QD}^2+\overline{BQ}\times\overline{QC}=\overline{BD}^2$ (*constr.* and E. 5. 2.)

To each of these equals add DA^2;

$\therefore \overline{AD}^2+\overline{QD}^2+\overline{BQ}\times\overline{QC}=\overline{AD}^2+\overline{DB}^2$:

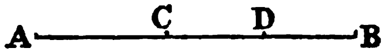
But (*constr.*) BD = DC, and DA is common to the △ ADB, ADC, and (*hyp.*) AB = AC; ∴ the ∠ ADB = ∠ ADC; and ∴ each of these ∠∠ is a right ∠; ∴ (E. 47. 1.) $\overline{AD}^2+\overline{DQ}^2=\overline{AQ}^2$, and $AD^2+DB^2=AB^2$;

$\therefore AQ^2+BQ\times QC=AB^2$.

Prop. IV.

5. Theorem. *The rectangle contained by the aggregate and the difference of two unequal straight lines is equal to the difference of their squares.*

Let AC and CB be two given unequal straight



lines, of which CB is the greater; and let them be placed in the same straight line AB; so that AB is the aggregate of AC, CB, and if (E. 3. 1.) CD be cut off from CB equal AC, DB is the difference between AC and CB. Then since (*constr.* and E. 6. 2.)

$$\overline{AB}\times\overline{DB}+\overline{AC}^2=\overline{CB}^2,$$

it is manifest, if from these equals AC^2 be taken, that $\overline{AB}\times\overline{DB}=\overline{CB}^2-\overline{AC}^2$;

i. e. the rectangle contained by the aggregate AB, of AC and CB, and their difference DB, is equal to the difference of their squares.

6. COR. If there be three straight lines, the difference between the first and second of which is equal to the difference between the second and third, the rectangle contained by the first and third, is less than the square of the second, by the square of the common difference between the lines.

For, let AB, CB, and DB be the three straight lines, having AC the difference of AB and CB, equal to CD, the difference of CB and DB: Then, since it has been shewn that $\overline{AB}\times\overline{DB}=\overline{CB}^2-\overline{AC}^2$, it is manifest that $\overline{AB}\times\overline{DB}$ is less than $\overline{CB}^2$ by $\overline{AC}^2$.

PROP. V.

7. THEOREM. *The square of the excess of the greater of two given straight lines above the less, is less than the squares of the two lines, by twice the rectangle contained by them.*

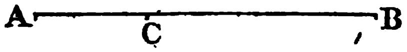
For let AB and CB be two given straight lines,

of which AB is the greater: Then is AC the excess of AB above CB; and since (E. 7. 2.) $\overline{AC}^2+2\overline{AB}\times\overline{BC}=\overline{AB}^2+\overline{CB}^2$, it is manifest that $\overline{AC}^2$ is less than $\overline{AB}^2+\overline{CB}^2$ by $2\overline{AB}\times\overline{BC}$.

PROP. VI.

8. **THEOREM.** *The squares of any two unequal straight lines are, together, greater than twice the rectangle contained by those lines.*

For let AB and CB be two given straight lines



of which AB is the greater: Then since (E. 7. 2.)

$$\overline{AB}^2+\overline{CB}^2=2\overline{AB}\times\overline{BC}+\overline{AC}^2$$

it is manifest that $\overline{AB}^2+\overline{CB}^2>2\overline{AB}\times\overline{BC}$.

PROP. VII.

9. **THEOREM.** *If a straight line be divided into five equal parts, the square of the whole line is equal to the square of the straight line, which is made up of four of those parts, together with the square of the straight line which is made up of three of those parts.*

Let the straight line AB be divided into five

equal parts by the points C, D, E, F: Then, $\overline{AF}^2+\overline{AE}^2=\overline{AB}^2$.

For since (*hyp.*) EF=FB ∴ $4\overline{FE}\times\overline{AF}+\overline{AE}^2=\overline{AB}^2$ (E. 8. 2.)

But, since AC=CD=DE=EF, 4FE=AF:

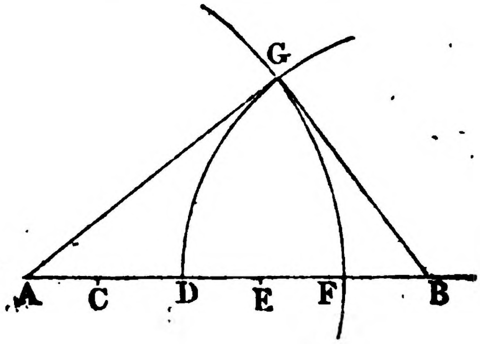
$$\therefore\ 4\overline{FE}\times\overline{AF}=\overline{AF}^2$$

$$\therefore\ \overline{AF}^2+\overline{AE}^2=\overline{AB}^2.$$

PROP. VIII.

10. PROBLEM. *Upon a given straight line, as an hypotenuse, to describe a right-angled triangle, such that the hypotenuse, together with the less of the two remaining sides, shall be the double of the greater of those sides.*

Let AB be the given straight line: Upon AB,



as an hypotenuse, it is required to describe a right-angled △, having the less of its two remaining sides, together with AB, the double of the third side.*

Divide (S. 49. 1.) AB into five equal parts in the points C, D, E, F; from A as a centre, at the distance AF, describe the circle FG, and from B as a centre at the distance BD, describe the circle DG cutting FG in G; join A, G and B, G: The △AGB is right-angled at G, and AB+BG is the double of AG.

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* That is, "Upon a given straight line, as an hypotenuse, to describe a right-angled triangle, the sides of which shall be arithmetic proportionals."

For (*constr.* and S. 7. 2.)

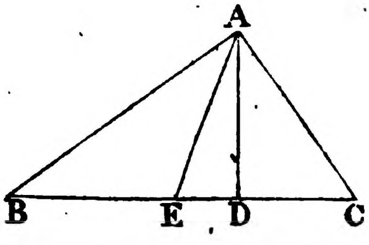
$$\overline{AB}^2 = \overline{AF}^2 + \overline{AE}^2$$
$$= \overline{AF}^2 + \overline{BD}^2 \text{ (constr.)}$$
$$= \overline{AG}^2 + \overline{BG}^2 \text{ (constr. and E. def. 15. 1.)}$$

Wherefore (E. 48. 1.) the △ AGB is right-angled at G: And since (*constr.*) AB, and BG, together contain eight of such equal parts as AG contains four, it is manifest that AB + BG is the double of AG.

Prop. IX.

11. Theorem. *In any triangle, the squares of the two sides are, together, the double of the squares of half the base, and of the straight line joining its bisection and the opposite angle.*

Let ABC be any given △, of which BC is the



base, and AE the straight line joining the vertex

A, and the bisection E of the base: Then, $\overline{AB}^2 + \overline{AC}^2 = 2\overline{AE}^2 + 2\overline{EB}^2$.

For from A draw (E. 12. 1.) AD ⊥ to BC, and first let AD fall within the base BC.

Then, $\overline{BD}^2 + \overline{DC}^2 = 2\overline{DE}^2 + 2\overline{EB}^2$. (E. 9. 2.)

Add to these equals $2\overline{AD}^2$.

$\therefore \overline{BD}^2 + \overline{DC}^2 + 2\overline{AD}^2 = 2\overline{AD}^2 + 2\overline{DE}^2 + 2\overline{EB}^2$.

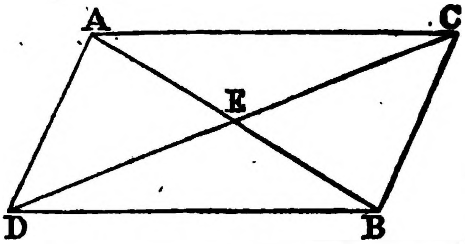
i. e. $\overline{AB}^2 + \overline{AC}^2 = 2\overline{AE}^2 + 2\overline{EB}^2$. (E. 47. 1.)

And, if the perpendicular AD fall without the base BC, the proposition may, in like manner, be deduced from E. 47. 1, and E. 10. 2.

PROP. X.

12. THEOREM. *The squares of the sides of any parallelogram are, together, equal to the squares of its diameters taken together.*

Let ACBD be a parallelogram, of which AB



and CD are the diameters: $\overline{AC}^2 + \overline{CB}^2 + \overline{BD}^2 + \overline{DA}^2 = \overline{AB}^2 + \overline{CD}^2$.

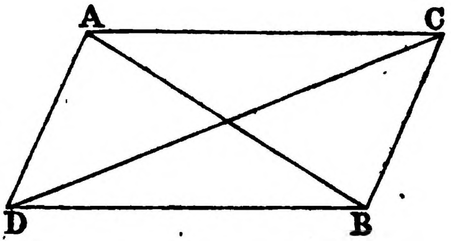
For (S. 42. 1.) AB and CD bisect one another in E:

$\therefore \overline{AC}^2 + \overline{CB}^2 + \overline{BD}^2 + \overline{DA}^2 = 2\overline{AE}^2 + 2\overline{DE}^2 + 2\overline{BE}^2 + 2\overline{DE}^2$ (S. 9. 2.) $= \overline{AB}^2 + \overline{CD}^2$ (E. 4. 2, and S. 42. 1.)

Prop. XI.

13. **Theorem.** *If either diameter of a parallelogram be equal to one of the sides about the opposite angle of the figure, its square shall be less than the square of the other diameter, by twice the square of the other side about that opposite angle.*

Let the diameter AB of the ▭ ACBD be equal



to one of the sides, as AC, about the opposite $\angle$ ACB; and let CD be the other diameter: Then $\overline{CD}^2 = \overline{AB}^2 + 2\overline{CB}^2$.

For, $\overline{CD}^2 + \overline{AB}^2 = 2\overline{AC}^2 + 2\overline{CB}^2$ (S. 10. 2, and E. 34. 1.)

From these equals take $\overline{AB}^2$ which (*hyp.*) is equal to $\overline{AC}^2$; and there remains,

$$\overline{CD}^2 = \overline{AC}^2 + 2\overline{CB}^2:$$

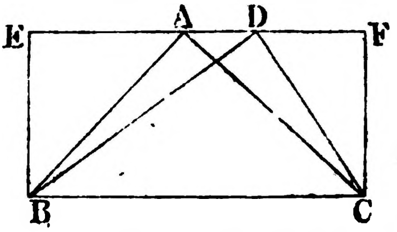
$$i.e.\ \overline{CD}^2 = \overline{AB}^2 + 2\overline{CB}^2:$$

Wherefore $\overline{AB}^2$ is less than $\overline{CD}^2$ by $2\overline{CB}^2$.

PROP. XII.

14. THEOREM. *If two sides of a trapezium be parallel to each other, the squares of its diagonals are, together, equal to the aggregate of the squares of its two sides, which are not parallel, and of twice the rectangle of its parallel sides.*

Let ABCD be a trapezium, having the side AD parallel to the side BC, and let $\overline{AC}$ and $\overline{BD}$ be its diameters; Then, $\overline{AC}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{DC}^2 + 2\overline{AD}\times\overline{BC}$.



From B and C, the extremities of BC, the greater of the two parallel sides, draw (E. 12. 1.) BE and CF, each ⊥ to AD; ∴ (*hyp.* and E. 28. 1.) the figure EBCF is a ▭, and (E. 34. 1.) EF $=$ BC.

First, let both the perpendiculars BE and CF fall without $\overline{AD}$, so that both of them meet $\overline{AD}$ produced.

$$\left.\begin{array}{l} \therefore \overline{AC}^2 = \overline{DC}^2 + \overline{AD}^2 + 2\overline{AD}\times\overline{DF} \\ \text{and}\,\overline{BD}^2 = \overline{AB}^2 + \overline{AD}^2 + 2\overline{AD}\times\overline{AE} \end{array}\right\} \text{(E. 12. 2.)}$$

$$\therefore \overline{AC}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{DC}^2 + 2\overline{AD}^2 + 2\overline{AD}\times\overline{AE} + 2\overline{AD}\times\overline{DF}.$$

But $2\overline{AD}^2+2\overline{AD}\times\overline{AE}+2\overline{AD}\times\overline{DF}=2\overline{AD}\times\overline{EF}$ (E. 1. 2.)

$\therefore \overline{AC}^2+\overline{BD}^2=\overline{AB}^2+\overline{DC}^2+2\overline{AD}\times\overline{EF}$:

$\therefore \overline{AC}^2+\overline{BD}^2=\overline{AB}^2+\overline{DC}^2+2\overline{AD}\times\overline{BC}$;

because, as hath been shewn, EF = BC.

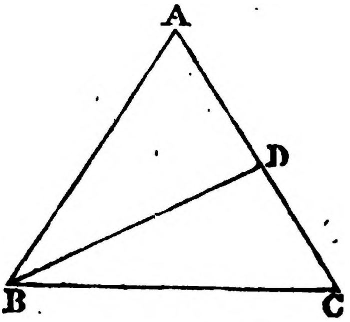
And, in like manner, may the proposition be demonstrated, by the help of E. 13. 2. if one of the perpendiculars drawn from B, C, fall within AD the less of the two parallel sides.

PROP. XIII.

15, THEOREM. *The square of the base of an isosceles triangle is the double of the rectangle contained by either side, and by the straight line intercepted between the perpendicular, let fall upon it from the opposite angle, and the extremity of the base.*

If the vertical angle of the isosceles △ be a right angle, the proof of the proposition is manifestly deducible from E. 47. 1.

But let ABC, be an isosceles △, having its ver-



tical ∠ A, not a right angle: First, let A be an acute ∠, and let BD be the perpendicular drawn from B to the opposite side AC: Then, $\overline{BC}^2 = 2\overline{AC} \times \overline{CD}$.

For, since BD is ⊥ to AC,

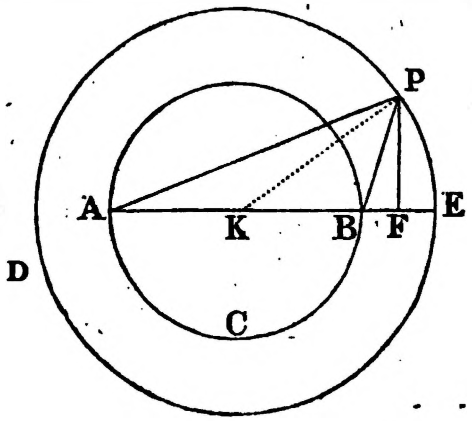
$$\therefore \overline{AB}^2 + 2\overline{AC} \times \overline{CD} = \overline{AC}^2 + \overline{BC}^2$$

From these equals take away the equal squares (*hyp.*) $\overline{AB}^2$ and $\overline{AC}^2$, and there remains $\overline{BC}^2 = 2\overline{AC} \times \overline{CD}$.

PROP. XIV.

16. THEOREM. *If from any point, in the circumference of the greater of two given concentric circles, two straight lines be drawn to the extremities of any diameter of the less, their squares shall be, together, the double of the squares of the two semi-diameters of the two given circles.*

Let ACB, PDE be two circles having a com-



mon centre K: and from any point P, in the circumference of the greater, let $\overline{PA}$, $\overline{PB}$, be drawn to the extremities A and B, of any diameter AKB, of the less circle: Then $\overline{PA}^2+\overline{PB}^2=2\overline{KA}^2+2\overline{KP}^2$, KA being a semidiameter of the less, and KP a semidiameter of the greater circle.

From P draw (E. 12. 1.) $\overline{PF} \perp$ to $\overline{AB}$; and, first, let $\overline{PF}$ fall without $\overline{AB}$. And, because $\overline{PF}$ is $\perp$ to $\overline{AB}$,

$$\therefore \overline{PB}^2+2\overline{BK}\times\overline{KF}=\overline{KP}^2+\overline{KB}^2 \text{ (E. 13. 2.)}$$

$$\text{also, } \overline{PA}^2 = \overline{KP}^2+\overline{KA}^2+2\overline{KA}\times\overline{KF};$$

wherefore, since (E. 15. def. 1.) $\overline{KB}=\overline{KA}$, if to the two former equals, the two latter be added, and if the equal rectangles, $2\overline{BK}\times\overline{KF}$, and $2\overline{KA}\times\overline{KF}$, be taken from the equal aggregates, it is manifest that

$$\overline{PA}^2+\overline{PB}^2=2\overline{KA}^2+2\overline{KP}^2.$$

And, in like manner, the proposition may be demonstrated, when the perpendicular PF falls within $\overline{AB}$, the diameter of the lesser circle.

A

SUPPLEMENT

TO THE

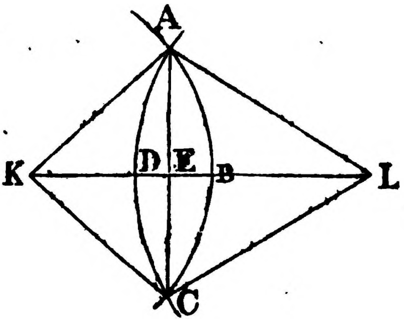
ELEMENTS OF EUCLID.

BOOK III.

PROP. I.

1. **THEOREM.** *If two circles cut each other, the straight line joining their two points of intersection is bisected, at right angles, by the straight line joining their centres.*

Let the circle ABC, of which the centre is K,



and the circle ADC, of which the centre is L, cut one another in the points A, C; and let K, L, and A, C be joined: $\overline{AC}$ is bisected, at right angles, in E, by $\overline{KL}$.

For, join K, A, and K, C, and L, A, and L, C: And since (E. def. 15. 1.) KA = KC and LA = LC, and that KL is common to the two △△ KAL, KCL, ∴ (E. 8. 1.) the ∠ AKL = ∠ CKL. Again, since AK = CK, and that KE is common to the two △△ AEK, CEK, and, as hath been shewn, the ∠ AKE = ∠ CKE, ∴ (E. 4. 1.) AE = CE, and the ∠ AEK = ∠ CEK, so that (E. def. 10. 1.) each of these ∠∠ is a right ∠. Wherefore, $\overline{KL}$ bisects AC at right angles.

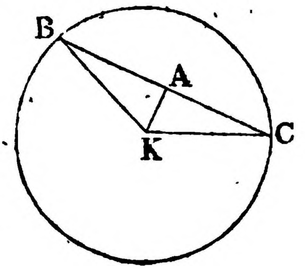
2. Cor. Hence, if a trapezium have two of its adjacent sides equal to one another, and also its two remaining sides equal to one another, its diameters bisect each other at right angles.

Prop. II.

3. Problem. *Through a given point within a circle, which is not the centre, to draw a chord which shall be bisected in that point.*

Let A be a given point within the circle BCD: It is required to draw, through A, a chord of the circle BCD, which shall be bisected in the point A.

Find (E. 1. 3.) the centre K of the circle BCD; join A, K; draw (E. 11. 1.) through A, the chord BAC ⊥ to KA: Then is BC bisected in the given point A.

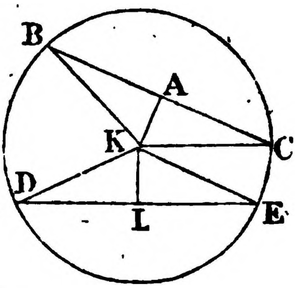


For (*constr.*) KA, which is drawn through the centre K, cuts BC at right angles in A; ∴(E. 3. 3.) BA = AC.

PROP. III.

4. THEOREM. *If two isosceles triangles be of equal altitudes, and the side of the one be equal to the side of the other, their bases shall be equal.*

Let BKC, DKE be two isosceles △, having



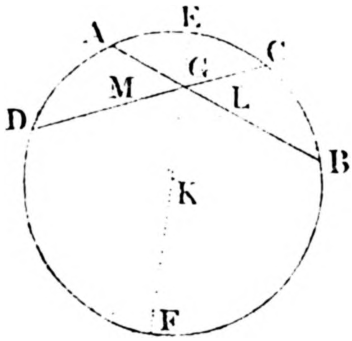
either of the equal sides, as BK, of the one, equal to either of the equal sides, as DK, of the other, and having, also, their altitudes, that is, the perpendiculars drawn the vertex to the base in each, equal to one another: The base BC is equal, also, to the base DE.

For the △ being supposed to be so placed as to have their summits in the same point K, if they wholly coincide, it is manifest, that $BC = DE$; And, if they do not coincide, since (*hyp.*) $KB = KD = KE = KC$, a circle described from the centre K, at the distance KB, will pass through D, E, and C. From K as a centre, describe, therefore, the circle BDEC; and since (*hyp.*) the $\perp KA = \perp KL$, $\therefore$ (E. 14. 3.) $BC = DE$.

Prop. IV.

5. Theorem. *Any two chords of a circle which cut a diameter in the same point and at equal angles, are equal to one another.*

Let any two chords AB, CD of the circle ADBC



cut a diameter EF in the same point G, and make with it the $\angle AGE = \angle CGE$: Then $AB = CD$.

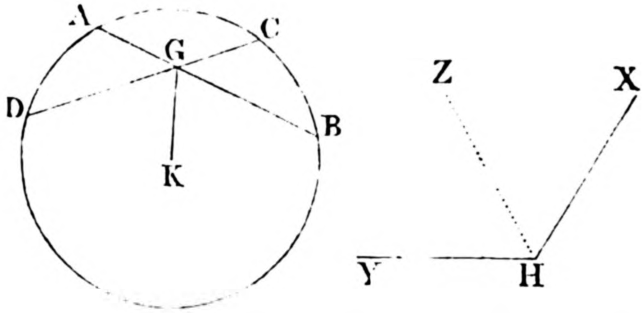
For, from K, the centre of the circle, draw

(E. 12. 1.) $\overline{KL} \perp$ to AB, and $\overline{KM} \perp$ to CD. And since (*hyp.* and E. 15. 1.) the $\angle LGK = \angle MGK$, and (*constr.*) the $\angle\angle$ at L and M are right angles, and that KG is common to the two $\triangle\triangle$ KLG, KMG, $\therefore$ (E. 26. 1.) $KL = KM$; $\therefore$ (E. 14. 3.) $AB = CD$.

PROP. V.

6. PROBLEM. *Through a given point, within a given circle, to draw two equal chords, making with one another an angle equal to a given rectilineal angle.*

Let G be a given point in the circle ADBC,



and XHY a given rectilineal angle: It is required to draw through G two equal chords of the circle ADBC, which shall make with one another an $\angle = \angle XHY$.

Find the centre K (E. 1. 3.) of the circle ADBC, and join K, G; bisect (E. 9. 1.) the $\angle XHY$ by $\overline{ZH}$; at the point G, in KG, make (E. 23. 1.) the $\angle\angle$ KGB, KGD each equal to the $\angle ZHX$ or ZHY, and produce BG and DG to meet the circumference in A and C respectively.

Then (*constr.*) the whole ∠BGD = ∠XHY; and since (*constr.*) the ∠KGB = ∠KGD, ∴ (S. 4. 3.) the chord AB = chord CD.

PROP. VI.

7. THEOREM. *If the diameters of two circles are in the same straight line, and have a common extremity, the two circles shall touch one another.*

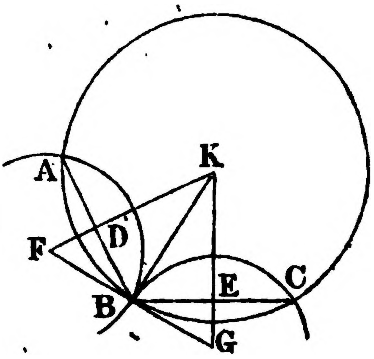
For since (*hyp.*) the two diameters are in the same straight line, it is manifest that a straight line drawn, from their common extremity, perpendicular to either of them will be perpendicular to the other, and ∴ (E. 16. 3. *cor.*) will touch both the circles: The circles, ∴, (E. 3. def. 3.) will touch one another: For it is plain that they cannot cut one another without also cutting the straight line that has been shewn to be their common tangent; which is impossible.

PROP. VII.

8. PROBLEM. *Three points being given in the circumference of a circle, and the middle point being equidistant from the other two, to describe two equal circles; which shall touch each other in the middle point, and which shall pass the one through*

one of the extreme points, and the other through the other extreme point.

Let A, B, C, be three given points in the cir-



cumference of the circle ABC, and let the middle point, B, be equidistant from A and B: It is required to describe two equal circles, the one passing through A and the other through C, which shall touch one another in B.

Join A, B, and B, C; find (E. 1. 3.) the centre K of the circle; from K draw (E. 12. 1.) $\overline{KD}$ ⊥ to $\overline{AB}$, and $\overline{KE}$ to $\overline{BC}$; join K, B; and through B draw (E. 11. 1.) $\overline{FBG}$ ⊥ to KB meeting KD and KE, produced in F and G respectively.

Then since (*hyp.*) AB = BC, ∴ (E. 14. 3.) KD = KE; and KB is common to the two △ KDB and KEB, and (*hyp. constr.* and E. 3. 3.) the side DB = the side EB; ∴ (E. 8. 1.) the ∠ DKB = ∠ EKB.

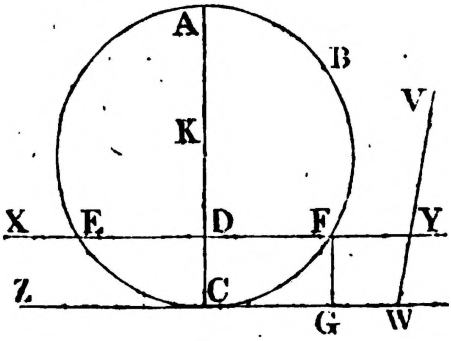
And, since KB is common to the two △ KBF, KBG, and the ∠ FKB = ∠ EKB, and (*constr.*) the

∠KBF = ∠KBG, ∴ (E. 26. 1.) FB = GB. If, ∴, from F and G, as centres, at the equal distances FB, GB, two equal circles be described, they will pass (*constr.* and S. 3. 1. cor. 2.) the one through A, and the other through B, and (S. 6. 3.) they will touch one another in the point B.

PROP. VIII.

9. PROBLEM. *To draw a tangent to a circle, which shall be parallel to a given finite straight line.*

Let ABC be a given circle, and XY a given straight line: It is required to draw a straight line which shall touch the circle ABC, and which shall be parallel to $\overline{XY}$.



Find (E. 1. 3.) the centre K of the circle ABC; from K draw (E. 12. 1.) the diameter AKC ⊥ to XY; and from either of the extremities, as C, of AC draw (E. 11. 1.) $\overline{ZCW}$ ⊥ to AC.

Then since ZCW is ⊥ to AC, at its extremity

C, it touches (E. 16. 3. *cor.*) the circle ABC: And since (*constr.*) the two ∠∠ XDC, DCZ are two right angles, ∴ (E. 28. 1.) ZW is parallel to XY.

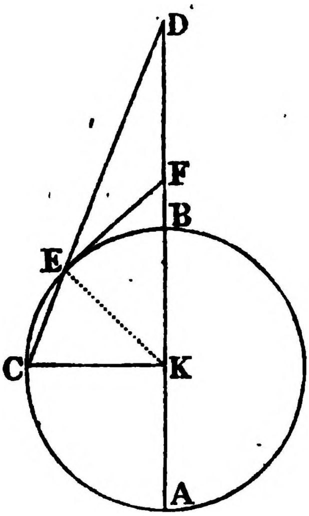
10. Cor. Hence a tangent may be drawn to a circle which shall make with a given straight line an ∠ equal to a given rectilineal angle.

For let it be required to draw a tangent to the circle ABC, which shall make with a given straight line VW an ∠ equal to a given ∠: Take any point Y in VW, and at the point Y, in $\overline{VY}$, make (E. 23. 1.) the ∠ VYX equal to the given ∠: If, then, the tangent ZW be drawn (S. 8. 3.) parallel to YX, it will make (E. 29. 1.) the ∠ ZWV = ∠ XYV, which is equal to the given ∠.

PROP. IX.

11. PROBLEM. *The diameter of a circle having been produced to a given point, to find in the part produced, a point from which, if a tangent be drawn to the circle, it shall be equal to the segment of the part produced, that is between the given point and the point found.*

Let the diameter AB of the circle ABC be produced to the given point D: It is required to find in BD a point from which if a tangent be drawn to the circle, it shall be equal to the part of BD which is between that point and D.



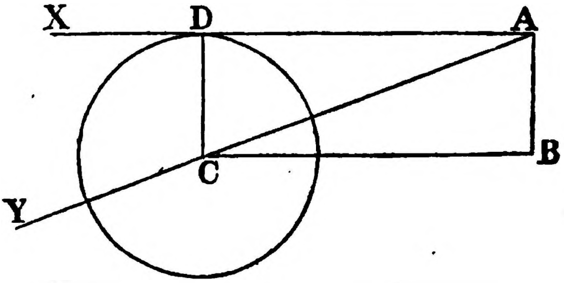
Find the centre K of the circle ABC; from K draw (E. 11. 1.) $\overline{KC} \perp$ to AB; join D, C, and let $\overline{DC}$ meet the circumference in E; join K, E; from E draw $\overline{EF} \perp$ to $\overline{KE}$ and let EF meet BD in F: Then is F the point which was to be found.

For (E. 13. 1.) the ∠∠ KEC, KEF, FED are together equal to two right angles; as are, also, (E. 32. 1.) the three ∠∠ DCK, CKD, and KDC, of the △ DKC: But since (E. 15. def. 1.) KE = KC, ∴ (E. 5. 1.) the ∠ KEC = ∠ KCE; and (*constr.*) the ∠∠ KEF, CKD are equal, each of them being a right angle; ∴ the remaining ∠ FED is equal to the remaining ∠ KDC or FDE: ∴ (E. 6. 1.) FE = FD; and since (*constr.*) EF is perpendicular to the semi-diameter KE, at its extremity E, ∴ (E. 16. 3.) FE touches the circle ABC. Q.E.F.

PROP. X.

12. **PROBLEM.** *To describe a circle which shall have a given semi-diameter and its centre in a given straight line, and shall also touch another straight line, inclined at a given angle to the former.*

Let AX and AY be two given straight lines inclined to one another at a given angle; and let L be a given finite straight line: It is required to describe a circle, which shall have its centre in AY, and its semi-diameter equal to L, and which shall touch AX.

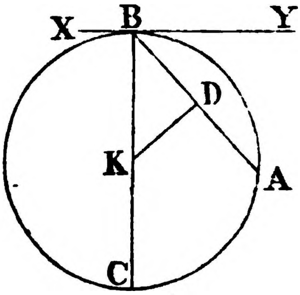


From the point A, in AX, draw (E. 11. 1.) AB ⊥ to AX, and make AB = L; through B draw (E. 31. 1.) BC parallel to AX, and through C draw CD parallel to AB: Wherefore, DB is a ▭; ∴ (E. 34. 1.) DC = AB; and since (*constr.*) the ∠ BAD is a right ∠; ∴ (E. 29. 1.) the ∠ ADC is also a right ∠: It is manifest, ∴, that a circle described from C as a centre, at the distance CD, will (E. 16. 3. *cor.*) touch AX; and its semi-diameter CD has been shewn to be equal to AB, which was made equal to the given straight line L. Q. E. F.

PROP. XI.

13. **PROBLEM.** *To describe a circle, the circumference of which shall pass through a given point, and touch a given straight line in another given point.*

Let B be a given point in the given straight line XY, and let A be any other given point, without that line: It is required to describe a circle the circumference of which shall pass through A and touch XY in B.



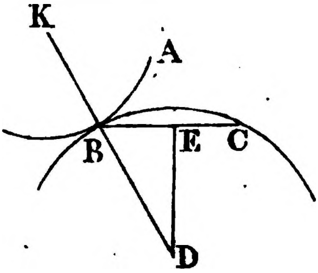
From B draw (E. 11. 1.) BC ⊥ to XY; join A, B; bisect (E. 10. 1.) $\overline{AB}$ in D, and from D draw $\overline{DK}$ ⊥ to AB; ∴ (S. 3. 1. cor. 2.) K is equidistant from A and B: It is manifest, therefore, that the circumference of a circle described from K as a centre, at the distance KB will pass through

A; and since BY (*constr.*) is ⊥ to KB, the circle so described will (E. 16. 3. *cor.*) touch XY in B.

PROP. XII.

14. PROBLEM. *To describe a circle, the circumference of which shall pass through a given point, and touch a given circle in another given point; the two points not lying in a tangent to the circle.*

Let B be a given point in the circle AB, and C



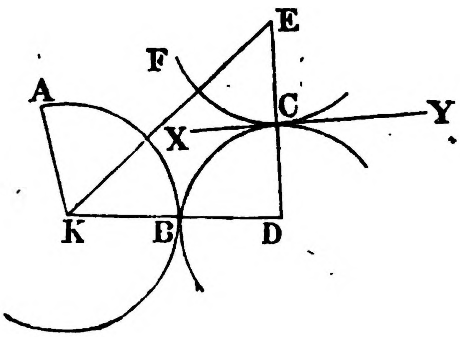
any other given point, which is not in a tangent to the circle at B: It is required to describe a circle, the circumference of which shall pass through C and touch the circle AB in B.

Find (E. 1. 3.) the centre K of the circle AB; join C, B and K, B; bisect (E. 10. 1.) CB in E; draw (E. 11. 1.) ED ⊥ to CB, and let ED meet KB, produced if necessary, in D: Then, since (S. 3. 1. cor. 2.) the point D is equidistant from B and C, the circumference of a circle described from D as a centre, at the distance DB, will pass through C, and (S. 6. 3.) it will touch the circle AB in B.

PROP. XIII.

15. **PROBLEM.** *To describe a circle, which shall touch a given straight line in a given point, and also touch a given circle.*

Let AB be a given circle, and let C be a given point in the given straight line XY: It is required to describe a circle which shall touch XY in C, and which shall also touch the circle AB.



Through C draw (E. 11. 1.) $\overline{\mathrm{ECD}} \perp$ to XY; find (E. 1. 3.) the centre K of the circle AB, and draw any semi-diameter of it at KA; make (E. 3. 1.) CE = KA, and join E, K; at the point K, in EK, make (E. 23. 1.) the ∠ EKD = ∠ KED, and let $\overline{\mathrm{KD}}$ meet $\overline{\mathrm{ECD}}$ in D: Then, since (*constr.*) the ∠ DEK = ∠ DKE, ∴ (E. 6. 1.) DE = DK; and CE = BK, for CE was made equal to KA, and (E. 15. def. 1.) KA = KB; ∴, the remainder DC is equal to the remainder DB; ∴, a circle described from D, as a centre, at the distance DC, will pass through B; and (E. 16. 3. *cor.* and *constr.*) it will touch $\overline{XY}$ in C, and (S. 6. 3.) it will also touch the circle AB in B.

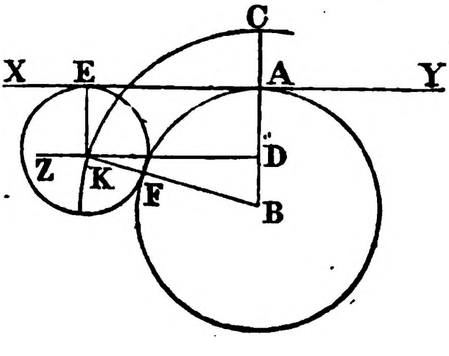
16. Cor. It is manifest that, in the same manner, a circle may be described which shall touch a given circle in a given point, and which shall, also, touch another given circle.

For, if a straight line be drawn at right angles to the diameter of the given circle that passes through the given point, that the solution of this latter problem is, evidently, reduced to that of the former.

Prop. XIV.

17. Problem. *To describe two circles, each having a given semi-diameter, which shall touch the same given straight line, both on the same side of it, and shall also touch each other.*

Let XY be a given straight line, of indefinite length: It is required to describe two circles, each



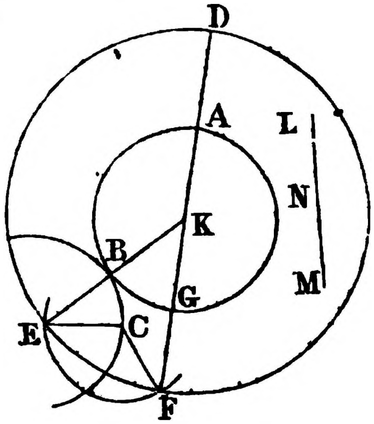
having a given semi-diameter, which shall touch $\overline{XY}$, and also touch one another.

Take any point A in XY; through A draw (E. 11. 1.) $\overline{CB}$ ⊥ to XY, and make AB equal to the given semi-diameter of one of the circles, and BC equal to the given semi-diameter of the other; from B, as a centre, at the distance BA describe the circle AF; from AB, produced, if necessary, cut off (E. 3. 1.) AD = AC; through D draw (E. 31. 1.) $\overline{DZ}$ parallel to XY; from B as a centre, at the distance BC describe a circle, and let its circumference cut $\overline{DZ}$ in K; through K draw KE parallel to AD, and join K, B; ∴ the figure AEKD is a ▭, and (E. 34. 1.) KE = DA or AC; also (*constr.* and E. 15. def. 1.) BC = BK; and BA = BF; ∴ the remainder AC = the remainder FK, and it has been shewn that AC = KE; ∴ KE = KF; ∴, a circle described from K as a centre, at the distance KE, will pass through F and (S. 6. 3.) will touch the circle AF, which circle (*constr.* and E. 16. 3. *cor.*) touches $\overline{XY}$; and since (*constr.*) the ∠ DAE is a right ∠, and that KE was drawn parallel to DA, ∴ (E. 29. 1.) the ∠ KEA is a right angle; ∴ (E. 16. 3. *cor.*) the circle EF, also, touches $\overline{XY}$; and its semi-diameter KE has been shewn to be equal to AD, which was made equal to the given semi-diameter. Q. E. F.

PROP. XV.

18. PROBLEM. *To describe two equal circles, each having its diameter equal to a given straight line, each touching a given circle, and each also passing through a given point without that circle: The given straight line being greater than the shortest distance, between the given point and the circumference of the given circle.*

Let ABG be a given circle, C a given point without it, and LM a given straight line greater than the shortest distance between C and the circumference of AB: It is required to describe two equal circles, each having its diameter = LM, each of them touching ABG and each passing through C.



Bisect (E. 10. 1.) LM in N; take any point A in $\widehat{ABG}$; find (E. 1. 3.) the centre K of ABG, and join K, A; produce KA to D and make AD = NL or NM; from K as a centre, at the distance KD, describe the circle DEF, and from the centre C at a distance equal to NL or NM, describe the circle EF and let $\overset{\frown}{EF}$ cut $\overset{\frown}{DEF}$ in the points E, and F; join E, K, and F, K, and let EK and FK meet $\overset{\frown}{ABG}$ in the points B and G; join, also, C, E and C, F.

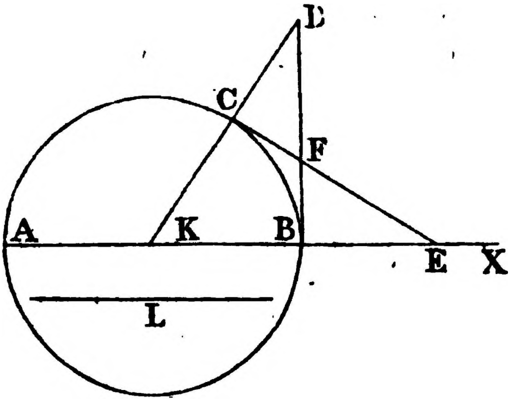
Then, it is manifest, (from the *constr.* and E. 15. def. 1.) that EB, EC, FC and FG are each of them equal to LN, the half of the given straight line LM; ∴, the two equal circles described from the centres E and F, at the equal distances EC and FC, will each of them have its diameter equal to LM, will each of them pass through the given point C, and (S. 6. 3.) will touch the given circle ABG in B and G.

19. Cor. In the same manner two equal circles may be described, each of them touching two given concentric circles, and each passing through a given point situated between the circumferences of those two given circles.

Prop. XVI.

20. Problem. *To find a point in the diameter, produced, of a given circle, from which, if a tangent be drawn to the circle, it shall be equal to a given straight line.*

Let AB be a diameter of the given circle ABC,



and let L be a given finite straight line: It is required to find a point in AB, produced, from which if a tangent be drawn to the circle ABC, it shall be equal to L.

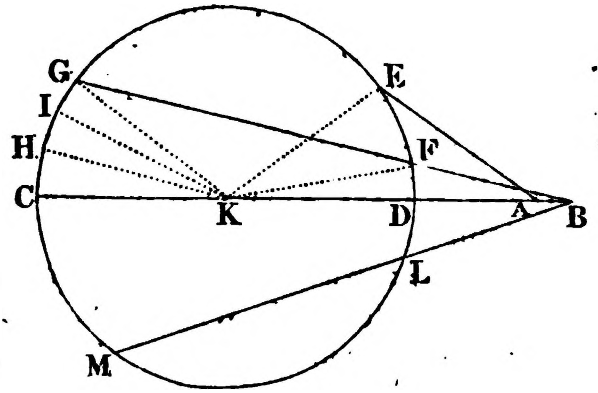
Produce AB, towards X; and from B draw (E. 11. 1.) $\overline{BD}$ ⊥ to $\overline{AB}$ and make $\overline{BD} = \overline{L}$; find (E. 1. 3.) the centre K of the circle ABC, and join K, D; let $\overline{KD}$ cut $\overset{\frown}{ABC}$ in C; from KX cut off (E. 3. 1.) KE = KD: Then is E the point which was to be found.

For, join E, C: And since (*constr.* and E. 15. def. 1.) the two sides DK, KB, are equal to the two sides EK, KC, and that the ∠ K is common to the two △ DBK, ECK, ∴ (E. 4. 1.) EC = BD and the ∠ ECK = ∠ DBK; but (*constr.*) BD = L, and the ∠ DBK is a right ∠; ∴ EC = L, and the ∠ ECK is a right ∠; ∴ (E. 16. 1. *cor.*) EC is a tangent to the circle ABC.

PROP. XVII.

21. THEOREM. *If the straight line, drawn from a point in the produced diameter of a circle to the convex circumference be equal to the half of the diameter, the angle at the centre, subtended by the* concave *circumference included between the diameter and the line so drawn, is the triple of the angle, at the centre, subtended by the* convex *circumference included between the same two lines.*

Let CDE be a given circle, of which K is the centre, and CDB, a produced diameter; and let $\overline{AE}$, which touches the circumference in E, or $\overline{BF}$, a part of $\overline{BFG}$, which cuts it, be equal to the semi-diameter of the circle: Then K, E, and K, F, and K, G, having been joined, the $\angle EKC = 3\angle EKD$; and the $\angle GKC = 3\angle FKD$.



For, first, since AE touches the circle, the ∠ AEK (E. 18. 3.) is a right ∠; ∴ (E. 32. 1.) the ∠ EAK + ∠ EKA = ∠ KEA; and (*hyp.*) EA = EK; ∴ (E. 5. 1.) the ∠ EAK = ∠ EKA; ∴ the ∠ KEA = 2 ∠ EKA; but (E. 32. 1.) the ∠ EKC = ∠ KEA + ∠ EKA; ∴ the ∠ EKC = 3 ∠ EKA.

Secondly, since (*hyp.* and E. 5. 1.) the ∠ FBK = ∠ FKB, and (E. 32. 1.) the ∠ GFK = ∠ FKB + ∠ FBK, ∴ the ∠ GFK = 2 ∠ FKB: But (E. 15. def. 1.) KF = KG; ∴ the ∠ KFG = ∠ KGF; ∴ the ∠ KGF = 2 ∠ FKD; and (E. 32. 1.) the ∠ GKC = ∠ KGB + ∠ GBK = ∠ KGF + ∠ FKD; ∴ the ∠ GKC = 3 ∠ FKD.

22. Cor. Hence, if a straight line could be drawn from any point in the curve of a semi-circle, to meet the diameter produced, so that the part of the line without the curve should be equal to the semi-diameter, any angle might be trisected.

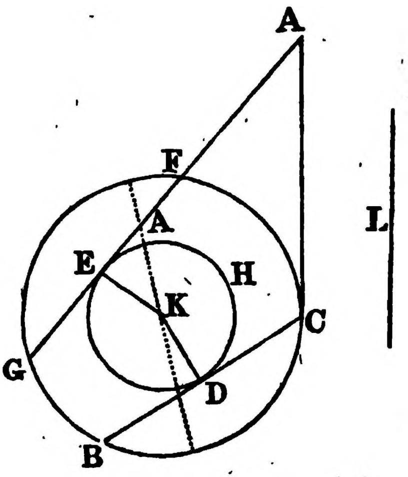
For, let GKC be any given angle; from K as a centre, at any distance KC, describe the circle CGD, and produce the diameter CD: Then, if from G, $\overline{\text{GFB}}$ could be drawn to meet CD produced in B, so that the part of it, FB, without the circle, should be equal to the semi-diameter KC, it is manifest from the proposition, that the ∠ GBC is the third part of the ∠ GKC: If, ∴, at the point K in $\overline{\text{CK}}$, the ∠ CKH were made (E. 23. 1.) equal to the ∠ CBG, and if, also, at the point K in $\overline{\text{HK}}$ the ∠ HKI were made equal to the same ∠ CBG, it is plain that the given ∠

GKC would thereby be divided into three equal parts.

PROP. XVIII.

23. PROBLEM. *Through a given point, either within, or without a given circle, to draw a straight line, so that the part of it within the circle shall be equal to a given finite straight line, which is not greater than the diameter.*

Let A be a given point, and L a given finite straight line, not greater than the diameter of FBC a given circle: It is required to draw through A a straight line cutting FBC, so that the part of it within the circle shall be equal to L.



Find (E. i. S.) the centre K of the given circle; and if L be equal to its diameter, let A, K be joined, and it is manifest that $\overline{AK}$ produced will be the straight line which was to be drawn.

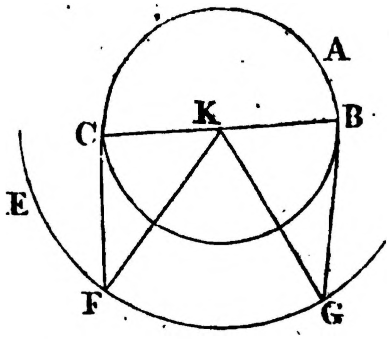
But if L be less than the diameter, take any point B in $\overset{\frown}{FGBC}$; from B as a centre, at a distance equal to L, describe a circle cutting $\overset{\frown}{FGBC}$ in C, and join B, C; ∴ (E. 15. def. 1.) BC = L: From K draw (E. 12. 1.) KD ⊥ to BC; from the centre K at the distance KD, describe the circle ED; from the point A draw (E. 17. 3.) $\overline{AE}$ touching the circle ED in E, and let AE, produced, meet the circumference in F and G: Then FG = L.

For (E. 15. def. 1.) KE = KD, and since AE touches the circle ED in E, the ∠ AEK is (E. 18. 3.) a right ∠, as is also (*constr.*) the ∠ KDB; ∴ (E. 14. 3.) FG = CB; but CB was made equal to L: ∴ FG = L.

Prop. XIX.

24. Theorem. *If, from any two points in the circumference of the greater of two given concentric circles, two straight lines be drawn so as to touch the less circle, they shall be equal to one another.*

Let F, G, be any two points in the circumference EFG of the greater of two circles, EFG, ABC, which have a common centre K: Two straight lines drawn from F and G so as to touch the less circle ABC shall be equal to one another.



For, draw (E. 17. 3.) from F and G, FC and $\overline{GB}$ touching ABC in C and B respectively; and join K, C and K, F and K, G and K, B; ∴ (E. 18. 3.) the ∠∠ KCF and KBG are right ∠∠; ∴ (E. 47. 1.)

$$\overline{KF}^2 = \overline{KC}^2 + \overline{CF}^2;$$

$$\text{and } \overline{KG}^2 = \overline{KB}^2 + \overline{BG}^2:$$

But (E. 15. def. 1.) KF = KG, and KC = KB; ∴ $\overline{KC}^2 + \overline{CF}^2 = \overline{KB}^2 + \overline{BG}^2$; take away the equal squares, $\overline{KC}^2$, and $\overline{KB}^2$, and there remains $\overline{CF}^2 = \overline{BG}^2$; ∴ CF = BG.

25. Cor. 1. In the same manner it may be shewn, that if two straight lines be drawn from any the same point so as to touch a given circle, they shall be equal to one another; and ∴, (E. 8. 1.) the straight line joining that point and the centre, bisects the ∠ contained by the two equal tangents.

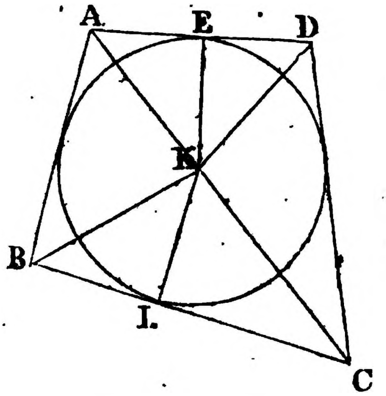
26. Cor. 2. If two circles touch one another and also touch a given straight line, which does not pass through their common point of contact,

a straight line that touches both the circles in their common point of contact shall bisect that other tangent straight line.

Prop. XX.

27. Theorem. *If a quadrilateral rectilineal figure be described about a circle, the angles subtended, at the centre of the circle, by any two opposite sides of the figure, are, together, equal to two right angles.*

Let the quadrilateral figure ABCD be described



about the circle EFLM, of which the centre is K; the ∠∠ subtended at K, by the two opposite sides AD, BC, or by AB, DC, are, together, equal to two right angles.

For join K, A, and K, B and K, C, and K, D: Then, because (E. 32. 1. *cor.* 1.) the four interior ∠∠ A, B, C, D, of the figure ABCD, are equal to four right ∠s, and that (S. 19. 3. *cor.* 1.) they are bisected by $\overline{KA}$, $\overline{KB}$, $\overline{KC}$ and $\overline{KD}$, respectively, ∴ the ∠s KAD, KDA, KBC, KCB are, together, equal to two right ∠s; but (E. 32. 1.) those ∠s, together with the ∠s AKD, BKC, being all the ∠s of the two △s AKD, BKC, are equal to four right ∠s; ∴ the ∠s AKD, BKC, are, together, equal to two right ∠s; ∴ (E. 15. 1. *cor.* 2.) the ∠s AKB, DKC, are, also, taken together, equal to two right angles.

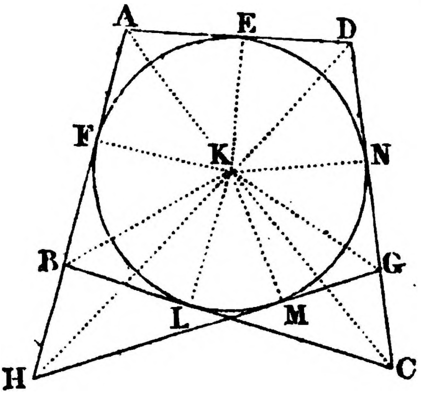
28. Cor. If two of the sides as AD, BC, of the quadrilateral figure described about the circle EFL, touch the circle at the extremities of a diameter, the ∠s subtended at the centre K, by each of the two remaining sides, shall be right angles.

For then since (E. 18. 3. and E. 28. 1.) $\overline{AD}$ is parallel to $\overline{BC}$; ∴ (E. 29. 1.) the ∠ EAB + ∠ ABL = two right angles; and (S. 19. 3. *cor.* 1.) the ∠ EAB is double of the ∠ BAK; in the same manner the ∠ ABL may be shewn to be double of the ∠ ABK; but it has been proved that the ∠ EAB + ∠ ABL = two right ∠s; ∴ the ∠ KAB + ∠ KBA = a right ∠; ∴ (E. 32. 1.) the ∠ AKB is a right ∠: But (S. 20. 3.) the ∠ AKB + ∠ DKC = two right ∠s; ∴ the ∠ DKC is, also, a right angle.

Prop. XXI.

29. Theorem. *If two given straight lines touch a circle, and if any number of other tangents be drawn, all on the same side of the centre, and all terminated by the two given tangents, the angles which they subtend, at the centre of the circle, shall be equal to one another.*

Let the two straight lines AH, DC touch the circle EFLM, and let $\overline{BC}$ and $\overline{GH}$ be any other tangents of the circle, both on the same side of the centre K, and both terminated by $\overline{AH}$ and $\overline{DC}$: Then $\overline{BC}$ and $\overline{GH}$ subtend equal ∠∠ at the centre.



For draw (E. 17. 3.) any other tangent to the circle, on the contrary side of the centre, as DEA, terminated in A and D, by AH and DC; and draw $\overline{KA}$, $\overline{KB}$, $\overline{KH}$, and $\overline{KC}$, $\overline{KG}$ and $\overline{KD}$: And because ABCD, AHGD, are quadrilateral figures described about the circle, ∴ (S. 20. 3.) the ∠ AKD + ∠ BKC = two right angles; and, the ∠ AKD + ∠ HKG = two right angles; ∴ the ∠ BKC = ∠ HKG; *i. e.* the ∠ subtended at the centre by the tangent BC is equal to the ∠ subtended at the centre by the tangent HG.

30. Cor. The two segments, which any two tangents, so drawn, cut off from the two given tangents, also subtend equal angles, at the centre of the circle.

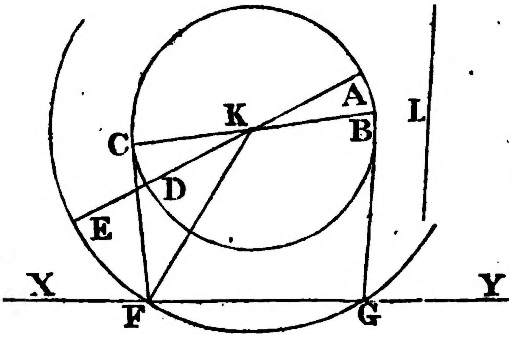
Let BH and GC be the segments cut off from the tangents AH and DC, by the two tangents BC and GH: They subtend equal ∠∠ BKH, GKC at the centre K.

For it has been shewn that the ∠ BKC = ∠ HKG; from these equals take away the common ∠ HKC, and there remains the ∠ BKH = ∠ GKC.

Prop. XXII.

31. Problem. *To draw a tangent to a given circle, such that its segment, contained between the point of contact, and an indefinite straight line, given in position, shall be equal to a given finite straight line.*

Let ABC be a given circle, L a given finite straight line, and XY an indefinite straight line



given in position: It is required to draw a tangent to ABC so that its segment between the point of contact and $\overline{XY}$ shall be equal to L.

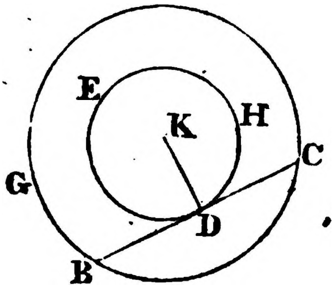
Find the centre K (E. 1. 3.) of the given circle, and take any diameter of it, as AKD; in AD produced find (S. 16. 3.) a point E from which if a tangent be drawn to the given circle ABC it shall be equal to L; from K as a centre, at the distance KE, describe the circle EFG, and let it meet, or cut, $\overline{XY}$ in F; from F draw (E. 17. 3.) $\overline{FC}$ to touch the circle ABC in C: And since (S. 19. 3.) FC is equal to the tangent which can be drawn from E, and which (*constr.*) is itself equal to L, it is manifest that CF = L; *i. e.* the segment of the tangent between the point of contact C and $\overline{XY}$ is equal to the given straight line L.

PROP. XXIII.

32. THEOREM. *If a straight line touch the interior of two concentric circles, and be terminated both*

ways by the circumference of the outer circle, it shall be bisected in the point of contact.

Let GBC, EDH be two circles having a common centre K, and let $\overline{BC}$ touch the interior circle EDH in D: Then is BC bisected in D.

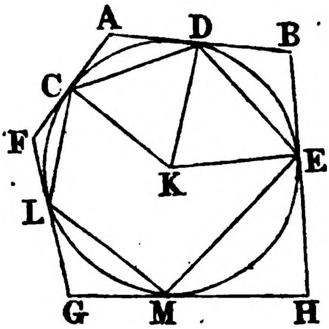


For join K, D: And, because BC touches EDH in D, the ∠∠ KDC, KDB (E. 18. 3.) are right ∠∠; ∴ (E. 3. 3.) $\overline{BC}$ is bisected in D.

Prop. XXIV.

33. Theorem. *If a polygon be described about a circle, the straight lines joining the several points of contact will contain a polygon of the same number of angles as the former; and any two adjacent angles of the circumscribed figure shall be, together, the double of that angle, of the inscribed figure, which lies between them.*

Let the sides of the polygon AFGHB touch the



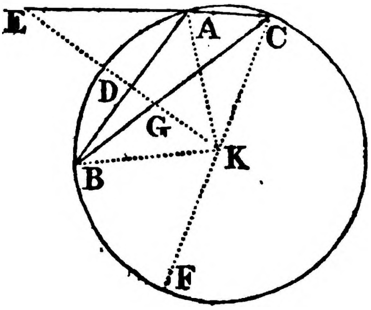
circle CLMED, in the several points C, L, M, E and D, and let these points be joined: Then it is manifest, that the polygon DCLME has the same number of angles as AFGHB; and, further, any two adjacent ∠∠ A and B of the polygon AFGHB, are, together, the double of the intermediate ∠ CDE, of the inscribed figure.

For, find (E. 1. 3.) the centre K of the circle DCLME, and join K, C and K, D and K, E: The four interior ∠∠ of the quadrilateral figure ACKD are (E. 32. 1. *cor.* 1.) equal to four right ∠∠; and (*hyp.* and E. 18. 3.) the ∠∠ ACK and ADK are right ∠∠; ∴ the ∠ DAC + ∠ CKD is equal to two right ∠∠, as are also (E. 32. 1.) the three ∠∠ of the isosceles △ CKD; ∴ ∠ DAC + ∠ CKD = ∠ DCK + ∠ CKD + ∠ KDC; take away the common ∠ CKD, and there remains the ∠ DAC equal to the two ∠∠ DCK, KDC, or to the double of the ∠ KDC; because (E. 15. def. 1. and 5. 1.) the ∠ DCK = ∠ KDC: And, in the same manner, it may be shewn that the ∠ B is the double of the ∠ KDE; ∴ the ∠ A + ∠ B is the double of the whole ∠ CDE.

PROP. XXV.

34. THEOREM. *If from any given point, in the circumference of a circle, two straight lines be drawn to the extremities of a given chord, the angle which the one makes with any perpendicular to the chord, shall be equal to the angle which the other makes with the diameter of the circle that passes through the given point.*

Let C be a given point in the circumference of the circle ABFC; let AB be a given chord; let C, A and C, B be joined; let K be the centre of the circle, and CKF a diameter passing through C; and let KD, drawn ⊥ to AB, meet CB in G, and CA, produced, in E: Then, the ∠ KEC = ∠ BCF, and the ∠ EGB = ∠ ECF.



For (E. 32. 1.) ∠ AEK + ∠ AKE = ∠ CAK:

But (*demonstr.* of E. 3. 3. and *constr.*)

the $\angle AKE = \frac{1}{2}\angle AKB$;

And (E. 20. 3.) the $\angle ACB = \frac{1}{2}\angle AKB$;

$\therefore \angle ACB = \angle AKE$.

Also (E. 15. def. 1. and E. 5. 1.)

$\angle CAK = \angle ACK$;

$\therefore \angle AEK + \angle ACB = \angle ACK$.

Take from both the $\angle ACB$ and there remains

$\angle AEK$ or $KEC = \angle BCF$.

Again (E. 32. 1.) the $\angle EGB = \angle ECG + \angle CEG$:

And it has been shewn that the $\angle CEG = \angle BCF$;

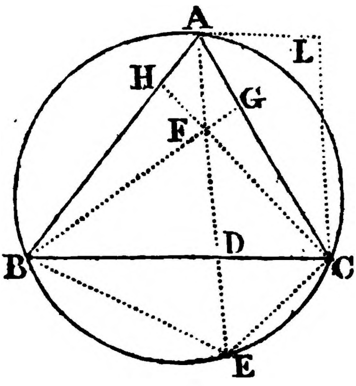
$\therefore \angle EGB = \angle ECB + \angle BCF$

i.e. $\angle EGB = \angle ECF$.

Prop. XXVI.

35. Theorem. *The perpendiculars let fall from the three angles of any triangle upon the opposite sides, intersect each other in the same point.*

Let ABC be a $\triangle$; the perpendiculars let fall



from the three ∠ A, B, C, on the sides opposite to them, intersect each other in the same point.

For draw (E. 12. 1.) $\overline{AD}$ ⊥ to BC; about the △ ABC describe (S. 5. 1. *cor.*) the circle ABC, and produce $\overline{AD}$ to meet the circumference in E; from DA, produced if necessary, cut off DF = DE; join B, E and C, E; also join B, F and C, F; and let BF and CF produced meet AC, and AB, in G and H respectively.

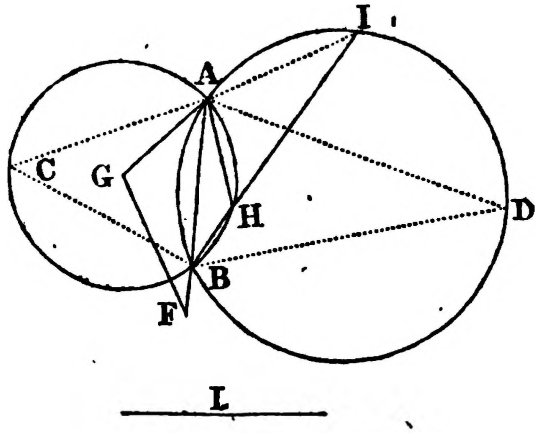
And since CD is common to the two △ CDF, CDE, and that (*constr.*) DF = DE, and the ∠ CDF = ∠ CDE, ∴ (E. 4. 1.) the ∠ FCD = ∠ DCE or BCE; but (E. 21. 3.) the ∠ BAE = ∠ BCE; ∴ the ∠ BAE or HAE = ∠ FCD; and (E. 15. 1.) the ∠ AFH, of the △ AHF, is also equal to the ∠ DFC, of the △ CDF; ∴ (S. 26. 1.) the ∠ AHF = ∠ FDC, which (*constr.*) is a right ∠; ∴ the ∠ CHA is a right ∠; *i. e.* CFH is ⊥ to AB; and, in the same manner, it may be shewn that $\overline{BFG}$ is ⊥ to AC: Whence it is manifest, that the three perpendiculars cut each other in the common point F; for (E. 17. 1.) there cannot be drawn, from the same point, two different straight lines both of them perpendicular to the same straight line.

36. Cor. The part of any of the three perpendiculars, let fall from the three ∠ of a △, on the opposite sides, that is, between their common intersection and the circumference of the circle described about the △, is bisected by the side to which it is perpendicular.

PROP. XXVII.

37. PROBLEM. *From either of the two given points in which two given circles intersect each other, to draw a chord cutting the one circumference, and meeting the other, such that the part of it, contained between the two circumferences, shall be equal to a given finite straight line.*

Let the two given circles ABC, ABD, cut one another in the points A and B, and let L be a given finite straight line: From either of the two given points, as B, it is required to draw a straight line cutting either of the circumferences, as that of ABC, and meeting the other circumference, so meeting that the part of it contained between the circumferences, shall be equal to L.



Take any point C in the circumference of ABC, and any point D in the circumference of ABD;

join A, B, and A, C, and A, D, and B, C and B, D; in AB, produced if necessary, take AF = L; at the point A, in AF, make (E. 23. 1.) the ∠ FAG = ∠ ACB, and at the point F, make the ∠ AFG = ∠ ADB, and let $\overline{AG}$ and $\overline{FG}$ meet in G. In the circle ABC place $\overline{AH}$ = AG; join B, H, and produce BH to meet the circumference of ADB in I: Then is HI = L.

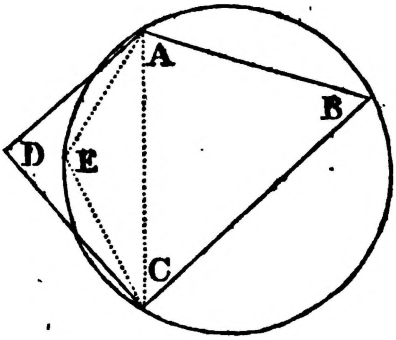
For, join A, I: And, since (E. 22. 3.) the ∠ AHB + ∠ ACB = two right ∠∠, and that (E. 13. 1.) the ∠ AHB + ∠ AHI = two right ∠∠, ∴ the ∠ AHI = ∠ ACB; but (*constr.*) the ∠ ACB = ∠ FAG; ∴ the ∠ AHI = ∠ FAG; and (*constr.*) the ∠ AFG = ∠ ADB, which (E. 21. 3.) = ∠ AIH; ∴ the ∠ AFG = ∠ AIB; and (*constr.*) the side AH, of the △ HAI, is equal to the side AG, of the △ AGF; ∴ (E. 26. 1.) HI = AF; and (*constr.*) AF = L; ∴ HI = L. Q. E. F.

PROP. XXVIII.

28. THEOREM. *If two opposite angles of a quadrilateral figure be together equal to two right angles, a circle may be described about it.*

Let any two opposite ∠∠, as the ∠∠ ABC, ADC, of the quadrilateral figure ABCD, be together equal to two right ∠∠: A circle may be described about the trapezium ABCD.

For, join A, C; and (S. 5. 1. *cor.*) about the △



ABC describe a circle: Its circumference shall pass through the point D. If not, let it pass otherwise, so that, first, the point D is without the circle ABC, described about the △ ABC; take any point E in the circumference of the circle and within the △ ADC; and join A, E and C, E: Then, since ABCE is a quadrilateral figure inscribed in a circle the ∠ ABC + ∠ AEC = two right ∠∠; and (*hyp.*) the ∠ ABC + ADC = two right ∠∠; ∴ the ∠ AEC = ∠ ADC, which (E.21.1.) is absurd. Wherefore the point D is not without the circle ABC; and in the same manner it may be shewn that the point D is not within the circle ABC; ∴, the circumference of the circle ABC passes through the point D, and is, ∴, a circle described about the four-sided figure ABCD.

PROP. XXIX.

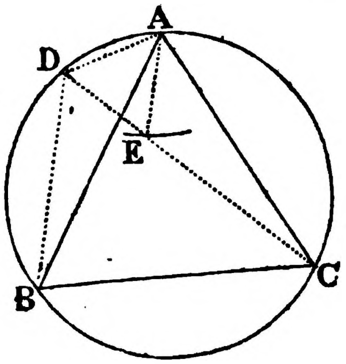
39. THEOREM. *A circle cannot be described about a rhombus, nor about any other parallelogram which is not rectangular.*

For (E. 34. 1.) the opposite ∠∠ of a ▭ are equal to one another; and (E. 22. 3.) if a circle could be described about it, the two opposite ∠∠ would, together, be equal to two right ∠∠; ∴, since these ∠∠ are equal, they would be each of them a right ∠; but (E. 32. def. 1.) the angles of a rhombus, which (E. 32. def. 1. and S. 18. 1.) is a ▭, are not right ∠∠; ∴ a circle cannot be described about a rhombus, nor about any other ▭, which has not its opposite ∠∠ right ∠∠, that is (S. 19. 1.) which is not rectangular.

Prop. XXX.

40. Theorem. *If from any point, in the circumference of a given circle, straight lines be drawn to the three angles of an inscribed equilateral triangle, the greatest of them shall be equal to the aggregate of the two less.*

Let the equilateral △ ABC be inscribed in the



circle ADBC, and from any point D in the circumference, let there be drawn to the three angular points A, B, C, the straight lines DA, DB, DC, of which DC is the greatest: Then DC $=$ DA $+$ DB.

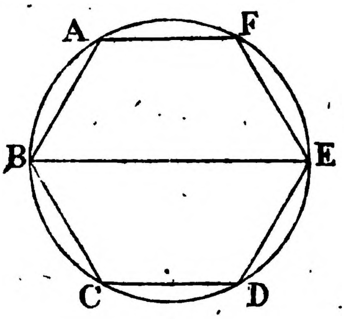
From the centre A, at the distance AD, describe a circle cutting DC in E, and join A, E; $\therefore$ (E. 15. def. 1.) AD $=$ AE; $\therefore$ (E. 5. 1.) the $\angle$ ADE $=$ $\angle$ AED; also (E. 21. 3.) the $\angle$ ADC or ADE $=$ $\angle$ ABC; and (*hyp.* and E. 5. 1.) the $\angle$ ABC $=$ $\angle$ ACB; $\therefore$ (S. 26. 1.) the $\angle$ DAE $=$ $\angle$ BAC; $\therefore$ the $\triangle$ ADE is equiangular; $\therefore$ (E. 6. 1.) AD $=$ DE.

Again, since (E. 22. 3.) the $\angle$ ACB $+$ $\angle$ ADB $=$ two right $\angle\angle$, and (E. 13. 1.) the $\angle$ AED $+$ $\angle$ AEC $=$ two right $\angle\angle$, and that the $\angle$ AED has been shewn to be equal to the $\angle$ ACB, $\therefore$ the $\angle$ AEC $=$ $\angle$ ADB; also (E. 21. 3.) the $\angle$ ACD or ACE $=$ $\angle$ ABD; and (*hyp.*) the side AC, of the $\triangle$ AEC, is equal to the side AB of the $\triangle$ ADB, $\therefore$ (E. 26. 1.) EC $=$ DB: And DA has been proved to be equal to DE; $\therefore$ DE $+$ EC $=$ DA $+$ DB; that is, DC $=$ DA $+$ DB.

PROP. XXXI.

41. THEOREM. *The first, third, fifth, &c. angles of any polygon, of an even number of sides, which is inscribed in a given circle, are together equal to the remaining angles of the figure; any angle whatever being assumed as the first.*

Let ABCDEF be any polygon, of an even number of sides, inscribed in the given circle ACE: Then A being assumed as the first $\angle$, the $\angle A + \angle C + \angle E +$, &c. $= \angle B + \angle D + \angle F +$, &c.



First, let the inscribed figure have six sides, and join B, E.

Then, since BAFE is a quadrilateral figure inscribed in a circle, ∴ (E. 22. 3.) the

$$\angle BAF + \angle FEB = \angle EFA + \angle EBA:$$

Also, the $\angle BCD + \angle BED = \angle EDC + \angle EBC$.

Wherefore, equals being added to equals, it will be manifest, that the $\angle BAF + \angle BCD + \angle FED = \angle CBA + \angle EDC + \angle AFE$:

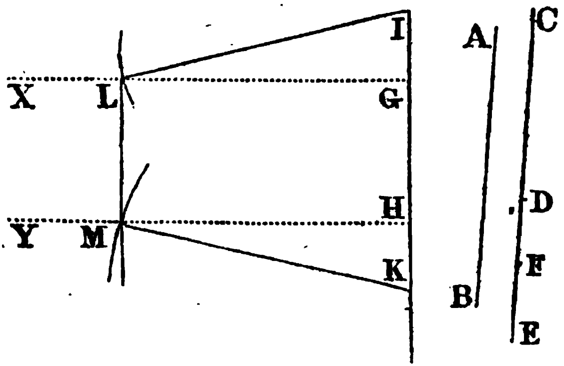
i. e. the $\angle A + \angle C + \angle E = \angle B + \angle D + \angle F$.

And, in a similar manner, the proposition may be demonstrated, when the figure inscribed in the given circle has eight, ten, twelve, or any other even number of sides.

Prop. XXXII.

42. **Problem.** *To make a trapezium, about which a circle may be described, having its four sides respectively equal to four given straight lines, two of which are equal to each other, and any three together greater than the fourth; the two equal sides of the trapezium, also, being opposite to each other.*

Let AB, CD, DE be three given straight lines:



It is required to make a trapezium having two of its opposite sides each of them equal to AB, and its two other sides equal to CD and CE, each to each, about which a circle may be described.

Take $\overline{GH}=\overline{CD}$; and CD and CE being placed in the same straight line, bisect (E. 10. 1.) DE in F; produce GH, both ways, and make GI and HK each of them equal to DF or FE; $\therefore$ IK $=$ CE: From the points G, H draw (E. 11. 1.) $\overline{GX}$

and HY $\perp$ to IK; from I and K, as centres, at distances equal to AB, describe two circles, cutting GX and HY in L and M, respectively; and join I, L and K, M; $\therefore$ (E. 15. def. 1.) $IL = AB$ and $KM = AB$; join L, M.

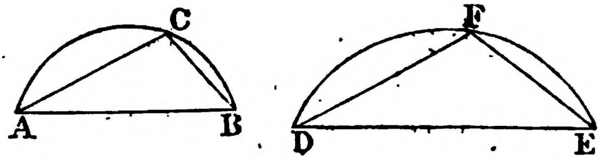
And, because (*constr.*) $LI = MK$, and $IG = KH$, and that the $\angle$s IGL, KHM, are right $\angle$s, (S. 74. 1.) $GL = HM$; and since, the $\angle$s at G and H are right $\angle$s, GL is (E. 28. 1.) parallel to HM; $\therefore$ (E. 33. 1.) LM is parallel and = to EH; but (*constr.*) $GH = CD$, $\therefore$ $LM = CD$.

Again, since GLMH is a ▭, the $\angle GLH = \angle GHM$ (E. 34. 1.) which (*constr.*) is a right $\angle$; also, since the two sides IL, LG, of the $\triangle$ LGI, are equal to the two sides KM, MH of the $\triangle$ MHK, and the base IG is equal to the base KH, $\therefore$ (E. 8. 1.) the $\angle ILG = \angle KMH$; but (*constr.* and E. 32. 1.) the $\angle HKM + \angle KMH =$ a right $\angle$; $\therefore$ the $\angle$ HKM, or IKM, $+$ ILG $=$ a right $\angle$; to each of these add the right $\angle$ GLM; $\therefore$ the $\angle IKM + \angle ILG + \angle GLM =$ two right $\angle$s; that is the $\angle IKM + \angle ILM =$ two right $\angle$s; $\therefore$ (S. 28. 3.) a circle may be described about the trapezium ILMK, which, as hath been shewn, has two equal sides LI, MK, each of them equal to AB, has its side LM equal to CD, and its remaining side IK equal to CG.

PROP. XXXIII.

43. PROBLEM. *Upon a given finite straight line to describe a segment of a circle, which shall be similar to a given segment of another circle.*

Let ACB be a given segment of a circle, and



DE a given finite straight line: It is required to describe on DE a segment of a circle, similar to the segment ACB.

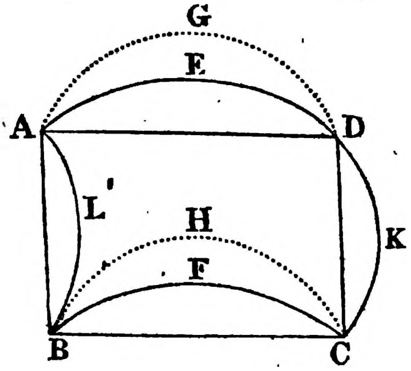
In $\overset{\frown}{ACB}$ take any point C, and join A, C, and B, C: At the point D in $\overline{DE}$ make (E. 23. 1.) the $\angle EDF = \angle BAC$; and, at the point E, also, make the $\angle DEF = \angle ABC$; $\therefore$ (S. 26. 1.) the $\angle DFE = \angle ACB$: About the $\triangle$ DFE describe (S. 5. 1. *cor.*) the circle DFE; $\therefore$ (E. 11. def. 3. and E. 21. 3.) the segment DFE is similar to the segment ACB.

PROP. XXXIV.

44. THEOREM. *If upon two opposite sides of an oblong, two similar segments of circles be described, the one of them lying wholly within the*

oblong, and the other wholly without it, the figure contained by the two remaining sides of the oblong and the two circular arches, shall be equal to the oblong.

Upon the two opposite sides AD, BC, of the



oblong ABCD, let there be described two similar segments of circles AED, BFC: the one, namely BFC, lying wholly within the oblong, and the other lying wholly without it: The figure contained by $\overline{\text{BA}}$, $\overset{\frown}{\text{AED}}$, $\overline{\text{DC}}$ and $\overset{\frown}{\text{CFB}}$ is equal to the oblong ABCD.

For (*hyp.* and E. 34. 1.) AD = BC; ∴ (*hyp.* and E. 24. 3.) the segment AED = the segment BFC; to each of these equals add the figure ADCFB, and it is manifest that the figure AEDCFB is equal to the oblong ABCD.

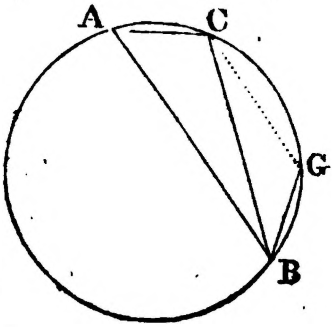
45. Cor. 1. An indefinite number of such mixtilineal figures may be found (S. 3. 1. *cor.* 3. and S. 33. 3.) equal to one another, and each of them equal to any given oblong.

46. Cor. 2. If upon AB and DC, the two remaining sides of the oblong, there be, likewise, described two similar segments of circles ALB, DKC, it is evident that the figure ALBCDA is equal to the figure ABFCDEA; and that the figure ALBFCKDE = ABCD, ALB being supposed not to meet BFC again within ABCD.

Prop. XXXV.

47. Theorem. *The arches of a circle that are intercepted between two parallel chords are equal to one another.*

Let AB and CG be two parallel chords of the circle ACGB: Then is $\overset{\frown}{AC} = \overset{\frown}{BG}$.

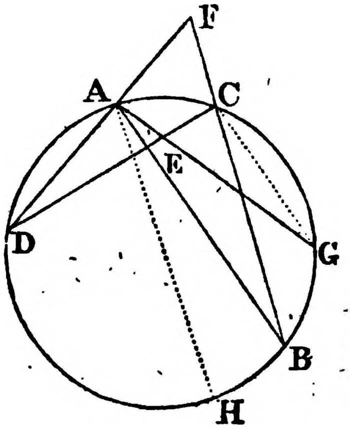


For join C, B: And, because (*hyp.*) CG is parallel to AB, ∴ (E. 29. 1.) the $\angle GCB = \angle CBA$; ∴ (E. 26. 3.) $\overset{\frown}{AC} = \overset{\frown}{BG}$.

Prop. XXXVI.

48. Theorem. *If two chords of a given circle intersect each other, the angle of their inclination is equal to the half of the angle at the centre standing upon the aggregate, or the difference, of the arches intercepted between them, accordingly as they meet within, or without the circle.*

First, let the two chords AB, CD, of the circle ACBD, cut one another in E, within the circle: The ∠ DEB is equal to the half of an angle at the centre, standing upon a circumference equal to $\overset{\frown}{AC} + \overset{\frown}{DB}$.



For through C draw (E. 31. 1.) CG parallel to AB; ∴ $\overset{\frown}{BG} = \overset{\frown}{AC}$, and $\overset{\frown}{DBG} = \overset{\frown}{AC} + \overset{\frown}{DB}$; but (*constr.* and E. 29. 1.) the ∠ DEB = ∠ DCG, which

(E. 20. 3.) is the half of an angle at the centre, standing upon $\overset{\frown}{DBG}$.

Secondly, let the two chords DA and BC, meet when produced, without the circle, in F: If, then, AH be drawn parallel to CB, it may be shewn, in a similar manner, that the ∠ DFB is equal to the half of an ∠ at the centre standing on $\overset{\frown}{DH}$, which is the difference between $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$.

PROP. XXXVII.

49. **THEOREM.** *In equal circles the greater angle stands upon the greater circumference; whether the angles compared be at the centres or the circumferences.*

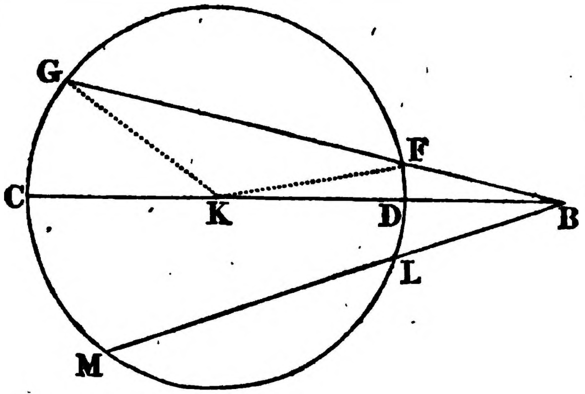
For whether the ∠∠ be at the centres, or the circumferences, if, from the greater, an ∠ (E. 23. 1.) be cut off equal to the less, the circumference on which it stands, will evidently be part of the circumference on which the greater ∠ stands, and will (E. 26. 3.) be equal to that on which the less ∠ stands; the which circumference is, ∴, less than the other.

PROP. XXXVIII.

50. **THEOREM.** *If from any given point, without a circle, there be drawn two straight lines cutting*

the circle, then of the circumferences which they intercept, that which is the nearer to the given point is less than the other.

From the given point B without the circle



FDCG, let there be drawn $\overline{BFG}$, $\overline{BDC}$, cutting the circumference in the points F, G, and D, C, respectively: Then is $\overset{\frown}{FD} < \overset{\frown}{GC}$.

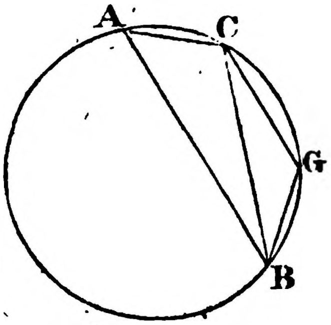
First, let one of the straight lines drawn from B, as BC, pass through the centre K of the circle: Join K, F and K, G; then (E. 16. 1.) the exterior $\angle GKC$, of the $\triangle GKB$, is > the $\angle KGF$; but (E. 15. def. 1. and E. 5. 1.) the $\angle KGF =$ $\angle KFG$; $\therefore$ the $\angle GKC >$ the $\angle KFG$; and (E. 16. 1.) the $\angle KFG >$ the $\angle FKB$ or FKD; much more, then, is the $\angle GKC > \angle FKD$; $\therefore$ (S. 37. 3.) $\overset{\frown}{CG} > \overset{\frown}{FD}$; *i. e.* $\overset{\frown}{FD} < \overset{\frown}{GC}$.

But if BLM do not pass through the centre, find (E. 1. 3.) the centre K; join B, K; and produce it to meet the circumference in C: Then it may be shewn, as before, that $\overset{\frown}{FD} < \overset{\frown}{GC}$, and that $\overset{\frown}{DL} < \overset{\frown}{CM}$; $\therefore$ $\overset{\frown}{FDL} < \overset{\frown}{GCM}$.

PROP. XXXIX.

51. THEOREM. *The straight lines joining the extremities of the chords of two equal arches of the same circle, toward the same parts, are parallel to each other.*

Let $\overline{AC}$, $\overline{BG}$ be the chords of two equal arches



$\overset{\frown}{AC}$, $\overset{\frown}{BG}$, of the circle ABGC; and let A, B, and C, G be joined: Then $\overline{CG}$ is parallel to $\overline{AB}$.

For join C, B; and since (*hyp.*) $\overset{\frown}{AC} = \overset{\frown}{BG}$, $\therefore$ (E. 27. 3.) the $\angle ABC = \angle BCG$; $\therefore$ (E. 27. 1.) $\overline{CG}$ is parallel to $\overline{AB}$.

PROP. XL.

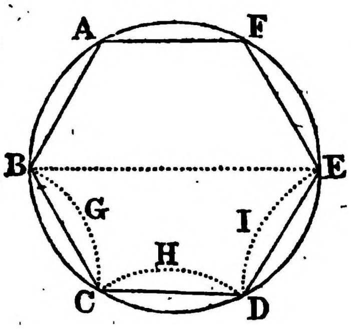
52. THEOREM. *In equal circles, the greater of two circumferences subtends the greater angle, whether the angles compared be at the centres or the circumferences.*

For if not, the ∠ standing on the greater circumference is equal to the other ∠ or less than it; but it cannot be equal; for then (E. 26. 3.) the two circumferences would be equal, which is contrary to the hypothesis: Neither can it be less, for, then, (S. 37. 3.) the greater circumference would be less than the other, which is absurd. Wherefore, the greater of two circumferences subtends the greater ∠, whether the two ∠∠ be at the centres or circumferences.

PROP. XLI.

53. PROBLEM. *If any equilateral rectilineal figure, of an* even *number of sides, be inscribed in a given circle, to find a curvilineal figure that is equal to it, and that is bounded by arches of circles, each of which circles is equal to the given circle.*

Let ABCDEF be an equilateral rectilineal figure, of an *even* number of sides, inscribed in the given circle ACE; It is required to find a



curvilineal figure equal to it, and bounded by arches of circles that are equal to the given circle ACE.

On half the number of sides of the inscribed figure, taken contiguous to one another, as BC, CD, DE, describe (S. 38. 3.) segments of circles, BGC, CHD, DIE, each similar to the segment cut off from the given circle by each of the sides: The curvilineal figure contained by $\overset{\frown}{BGC}$, $\overset{\frown}{CHD}$, $\overset{\frown}{DIE}$, $\overset{\frown}{EF}$, and $\overset{\frown}{FA}$, and $\overset{\frown}{AB}$, is equal to the inscribed polygon ABCDEF.

For, since (*hyp.*) the figure is equilateral, (E. 28. 3.) $\overset{\frown}{AB} = \overset{\frown}{BC}$; ∴ $\overset{\frown}{AFEDCB} = \overset{\frown}{BAFEDC}$; ∴ (E. 27. 3.) the ∠ in the segment cut off by $\overline{AB}$ is equal to the ∠ in the segment cut off by $\overline{BC}$; ∴ these two segments (E. 11. def. 3.) are similar, and (*hyp.* and E. 24. 3.) equal to one another.

And, in the same manner, may all the segments, cut off by the equal sides of the inscribed figure, be shewn to be similar and equal to one another, and to the segments BGC, CHD, DIE.

But the figure contained by $\overline{BA}$, $\overline{AF}$, $\overline{FE}$, $\overset{\frown}{EID}$, $\overset{\frown}{DHC}$ and $\overset{\frown}{CGB}$, together with the segments BGC, CHD, DIE, makes up the equilateral rectilineal figure ABCDEF; and that same figure, together with the equal segments cut off by BA, AF, and FE, makes up the curvilineal figure contained by $\overset{\frown}{BGC}$, $\overset{\frown}{CHD}$, $\overset{\frown}{DIE}$, $\overset{\frown}{EF}$, $\overset{\frown}{FA}$ and $\overset{\frown}{AB}$; the which figure is, ∴, equal to the inscribed rectilineal figure ABCDEF.*

Prop. XLII.

54. Theorem. *In equal circles, the greater chord subtends the greater circumference.*

For (*hyp.* and E. 15. def. 1. and E. 25. 1.) the ∠ subtended, at the centre, by the greater chord is > the ∠ subtended, at the centre, by the less: ∴ (S. 37. 3.) the circumference subtended by the greater chord is greater than the circumference subtended by the less.

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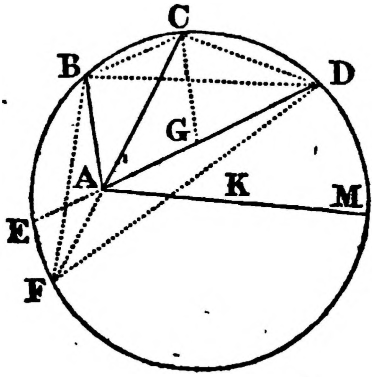
* It is easy to shew, by the help, chiefly, of S. 7. 3., that when the equilateral figure inscribed in the circle, is a square, the circumferences of the similar segments, described in the course of the demonstration, touch one another in the common extremity of the two contiguous sides; and that when the inscribed polygon has any greater number of sides, as six, eight, &c., the circumferences of any two of the segments meeting one another in the common extremity of two contiguous sides, do not meet again within the circle.

Prop. XLIII.

55. Theorem. *If from a given point within a circle, which is not the centre, straight lines be drawn to the circumference, making with each other equal angles, the two, which are nearer to the diameter passing through the given point, shall cut off a greater circumference than the two, which are more remote.*

From A, a given point within the circle BDC,



let there be drawn to the circumference any number of straight lines AB, AC, AD, &c. containing equal ∠ BAC, CAD, &c.; and let $\overline{AKM}$ be drawn, from A, through the centre K: Then is $\overset{\frown}{CD} > \overset{\frown}{CB}$.

For produce DA and CA, to meet the circumference again, in E and F; and join B, F and D, F: Then, since AD is nearer than AB is to AM, $\therefore$ (E. 7. 3.) $AD > AB$; from AD cut off $AG = AB$, and join C, G and C, B, and C, D and D, B: And, because CA is common to the two $\triangle$s CBA, CGA, and $AB = AG$, and that (*hyp.*) the $\angle CAB = \angle CAG$, $\therefore$ $CG = CB$. Again, because (E. 32. 1.) the $\angle CGD = \angle GCA + \angle CAG = \angle ACB + \angle BAC$, and that (E. 16. 1.) the $\angle BAC > \angle BFC$, $\therefore$ the $\angle CGD > \angle FCB + \angle BFC$: But (E. 21. 3.) the $\angle FCB = \angle FDB$, and the $\angle BFC = \angle BDC$; $\therefore$ the $\angle CGD > \angle FDB + \angle BDC$; *i. e.* the $\angle CGD > \angle FDC$, much more then is the $\angle CGD > \angle EDC$ or $\angle GDC$; $\therefore$ (E. 19. 1.) $CD > CG$; but it has been shewn that $CG = CB$; $\therefore$ $\overline{CD} > \overline{CB}$; $\therefore$ (S. 42. 3.) $\overset{\frown}{CD} > \overset{\frown}{CB}$.

PROP. XLIV.

56. THEOREM. *In equal circles, the greater circumference has the greater chord.*

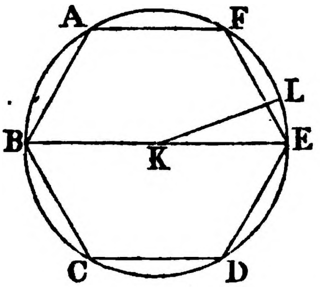
For (S. 40. 3.) the greater circumference subtends the greater $\angle$ at the centre; $\therefore$ (E. 15. def. 1. and E. 24. 1.) it has the greater chord.

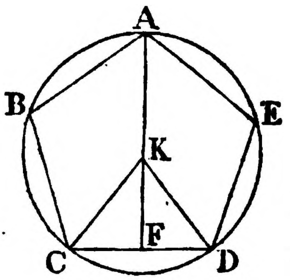
PROP. XLV.

57. THEOREM. *The straight line joining any of the angular points of an equilateral polygon inscribed*

in a circle and the centre, passes through the opposite angular point, or else bisects the opposit side at right angles, accordingly as the figure has an even, *or an* odd *number of sides.*

First, let the equilateral polygon ABCDEF in-





scribed in the circle ACE, of which the centre is K, have an even number of sides, and let B, K be joined, B being any one of the angular points of the inscribed figure: Then $\overline{\mathrm{BK}}$ passes through the opposite angular point E.

For, if it be possible, let BK cut the circumference in any other point L; ∴ (E. 28. 3.) $\overset{\frown}{\mathrm{BAL}}$ is the half of the whole circumference; also, since the polygon (*hyp.*) is equilateral, the arches $\overset{\frown}{\mathrm{AB}}$, $\overset{\frown}{\mathrm{BC}}$, $\overset{\frown}{\mathrm{CD}}$, $\overset{\frown}{\mathrm{DE}}$, $\overset{\frown}{\mathrm{EF}}$, $\overset{\frown}{\mathrm{FA}}$ are (E. 28. 3.) equal to one another; ∴ $\overset{\frown}{\mathrm{BAE}}$ is the half of the whole circumference; ∴ $\overset{\frown}{\mathrm{BAL}} = \overset{\frown}{\mathrm{BAE}}$, the less to the greater, which is absurd: ∴ $\overline{\mathrm{BK}}$, produced, passes through E.

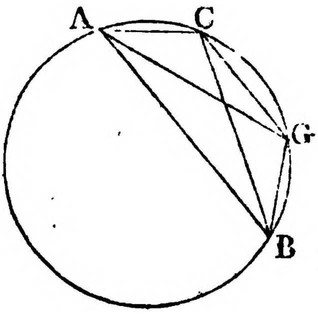
But, secondly, let the equilateral polygon ABCDE, inscribed in the circle ACE, have an odd number of sides, and let any of the angular points, as A, and the centre K, be joined: Then $\overline{AK}$, produced, bisects CD, at right $\angle\angle$ in the point F.

For join K, C and K, D: And, because (*hyp.*) the sides of the inscribed polygon are equal, the circumferences which they subtend are (E. 28. 3.) equal; since, therefore, the polygon has an odd number of sides, it is manifest that the circumference ABC is equal to the circumference AED; $\therefore$ (E. 27. 3.) the $\angle AKC = \angle AKD$; $\therefore$ (E. 13. 1.) the $\angle CKF = \angle DKF$; and (E. 15. def. 1.) $CK = DK$, and KF is common to the two $\triangle\triangle$ KFC, KFD; $\therefore$ (E. 4. 1.) $CF = FD$, and the $\angle KFC = \angle KFD$; so that $\overline{AKF}$ bisects at right $\angle\angle$ the side CD, which is opposite to the $\angle BAE$.

PROP. XLVI.

58. THEOREM. *The two straight lines in a circle, which join the extremities of two parallel chords, are equal to each other.*

Let $\overline{AB}$, $\overline{CG}$ be two parallel chords, of the circle ABGC, and let their extremities be joined, toward the same parts by $\overline{CA}$ and $\overline{GB}$, and towards opposite parts by $\overline{CB}$ and $\overline{GA}$: Then $\overline{CA} = \overline{GB}$, and $\overline{CB} = \overline{GA}$.

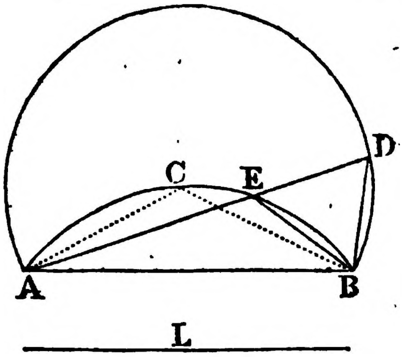


For, since $\overline{CG}$ is parallel to $\overline{AB}$, the arch $\overset{\frown}{CA} = \overset{\frown}{GB}$ (S. 35. 3.); ∴ (E. 29. 3.) $\overline{CA} = \overline{GB}$: Again, since it has been shewn that $\overset{\frown}{AC} = \overset{\frown}{BG}$, to each of these add $\overset{\frown}{CG}$; ∴ $\overset{\frown}{ACG} = \overset{\frown}{BGC}$; ∴ (E. 29. 3.) $\overline{GA} = \overline{CB}$.

PROP. XLVII.

59. PROBLEM. *To divide a given circular arch into two parts, so that the aggregate of their chords may be equal to a given straight line, greater than the chord of the whole arch, but not greater than the double of the chord of half the arch.*

Let $\overset{\frown}{ACB}$ be a given circular arch, of which $\overline{AB}$ is the chord; and let L be a given finite straight line, greater than AB, but not greater than twice



the chord of the half of $\overset{\frown}{ACB}$: It is required to divide $\overset{\frown}{ACB}$ into two parts such that the aggregate of their chords shall be equal to L.

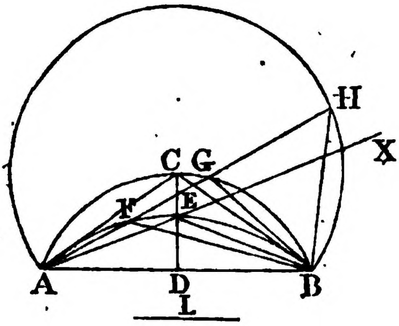
Bisect (E. 30. 3.) $\overset{\frown}{ACB}$ in C, and join C, A, and C, B; ∴ (E. 29. 3.) $\overline{CA}=\overline{CB}$; from the centre C, at the distance CA, or CB, describe the circle ADB, which will, ∴, pass through both A and B: From the centre A, at a distance equal to L, describe a circle, cutting the circle ADB in D; draw $\overline{AD}$, which is, ∴, equal to L; let $\overline{AD}$ cut $\overset{\frown}{ACB}$ in E, and join E, B: Then $\overline{AE}+\overline{EB}=L$.

For (E. 20. 3.) the ∠ ACB is the double of the ∠ ADB; and (E. 21. 3.) the ∠ ACB = ∠ AEB; ∴ the ∠ AEB is the double of the ∠ ADB; but (E. 32. 1.) the ∠ AEB = ∠ EDB + ∠ EBD; ∴ the ∠ EDB + ∠ EBD is equal to the double of the ∠ EDB; from these equals take the ∠ EDB, and there remains the ∠ EBD = ∠ EDB; ∴ (E. 6. 1.) ED = EB; ∴ AE + EB = AE + ED or AD; but (*constr.*) AD = L; ∴ AE + EB = L.

PROP. XLVIII.

60. PROBLEM. *To divide a given circular arch into two parts, so that the excess of the chord of the one above the chord of the other, may be equal to a given straight line, less than the chord of the whole arch.*

Let ACB be a given circular arch, of which the



chord is AB: It is required to divide $\overset{\frown}{ACB}$ into two parts, such that the excess of the chord of the one above the chord of the other shall be equal to a given finite straight line L, that is less than AB.

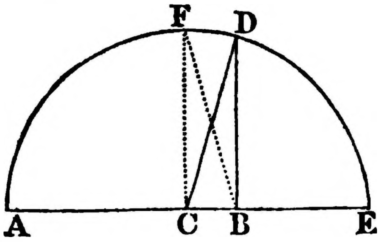
Bisect AB (E. 10. 1.) in D; draw $\overline{DC}$ (E. 11. 1.) ⊥ to DB; join A, C; bisect (E. 9. 1.) the ∠ CAB by $\overline{AX}$; let AX meet CD in E; and join $\overset{\frown}{B, E}$; about the △ AEB describe (S. 5. 1. *cor.*) the circle AEB; from the centre A, at a distance = L, describe a circle cutting $\overset{\frown}{AEB}$ in F; draw

$\overline{AF}$, which is, ∴, equal to L, and produce $\overline{AF}$ to meet $\overset{\frown}{ACB}$ in G; join G, B: Then is the excess of AG above GB equal to L.

For join C, B; and (*constr.* and E. 4. 1.) CA = CB and the ∠ACD = ∠BCD; also EA = EB, and the ∠ ACB is the double of the ∠ACD; Again, from the centre C at the distance CA, describe the circle AHB, which, because CA = CB passes through B; produce $\overline{AG}$ to meet $\overset{\frown}{AHB}$ in H; join H, B, and F, B: And since (E. 21. 3.) the ∠AFB = ∠AEB, ∴ (E. 13. 1.) the ∠BFH = ∠BEX; but (E. 32. 1. and E. 5. 1.) the ∠BEX is the double of the ∠BAE; ∴ the ∠BFH is the double of the ∠BAE, and is ∴ (*constr.*) equal to the ∠CAB or CAD; also (E. 20. 3. and *constr.*) the ∠ACB is the double of the ∠AHB; and it is also, as hath been shewn, the double of the ∠ACD; ∴ the ∠ACD = ∠AHB or FHB; and it has been proved that the ∠HFB, of the △FBH, is equal to the ∠CAD of the △CDA; ∴ (S. 26. 1.) the ∠HBF = ∠CDA, and is, ∴, a right ∠; but (*demonstr.* of S. 47. 3.) GH = GB; ∴ (S. 29. 1. *cor.* 3.) GF = GB; ∴ AG − GB = AF; but (*constr.*) AF = L; ∴ CAG − GB = L.

PROP. XLIX.

61. THEOREM. *If from any point, in the diameter of a semi-circle, there be drawn two straight lines to the circumference, one to the bisection of the circumference, the other at right angles to the diameter, the squares upon these two lines are, together, the double of the square upon the semi-diameter.*



Let B be any point in the diameter AE of the semi-circle ADE; let F be the bisection of the circumference ADE; and let C be the bisection of the diameter: If B, F be joined, and BD be drawn ⊥ to AE, then $\overline{BF}^2 + \overline{BD}^2 = 2\overline{AC}^2$.

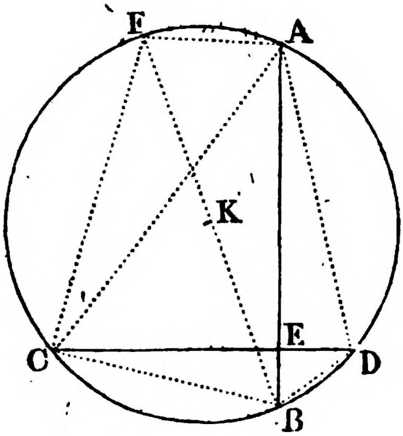
For join C, F and C, D: And since (*hyp.*) $\overset{\frown}{AF} = \overset{\frown}{EF}$ ∴ (E. 27. 3.) the $\angle ACF = \angle ECF$; and they are adjacent $\angle\angle$; ∴ (E. 10. def. 1.) the $\angle FCE$ is a right $\angle$; ∴ (E. 47. 1.) $\overline{BF}^2 = \overline{CF}^2 + \overline{CB}^2$; to each of these add $\overline{BD}^2$; ∴ $\overline{BF}^2 + \overline{BD}^2 = \overline{CF}^2 + \overline{CB}^2 + \overline{BD}^2$; but (*hyp.* and E. 47. 1.) $\overline{CB}^2 + \overline{BD}^2 =$

$\overline{CD}^2$; $\therefore \overline{BF}^2+\overline{BD}^2=\overline{CF}^2+\overline{CD}^2$ or (E. 15. def. 1.) $2\overline{AC}^2$.

Prop. L.

62. Theorem. *If the chords of two arches of any the same circle cut each other at right angles, the squares of the four segments of the chords, are, together, equal to the square of the diameter.*

Let the two chords AB, CD of the circle ACD, cut each other at right ∠∠, in E: The squares of the segments of the chords are, together, equal to the square of the diameter of the circle.



For find (E. 1. 3.) the centre K, and from either extremity of either of the chords, as B, draw through K the diameter BKF; join B, C and C, F, and F, A and A, D. And since (*constr.*) FADB is a semicircle, the ∠ FAB is (E. 31. 3.)

a right $\angle$, as is, also, (*hyp.*) the $\angle$AEC; $\therefore$ (E. 28. 1.) FA is parallel to CD; $\therefore$ (S. 44. 3.) $\overline{FC}=\overline{AD}$. Again, because FCB is a semi-circle, the $\angle$ BCF (E. 31. 3.) is a right $\angle$; $\therefore$ (E. 47. 1.) $\overline{FB}^2=\overline{FC}^2+\overline{CB}^2$; but FC has been shewn to be equal to AD $\therefore$ $\overline{FB}^2=\overline{AD}^2+\overline{CB}^2$; that is (*hyp.* and E. 47. 1.) $\overline{FB}^2=\overline{AE}^2+\overline{ED}^2+\overline{CE}^2+\overline{EB}^2$.

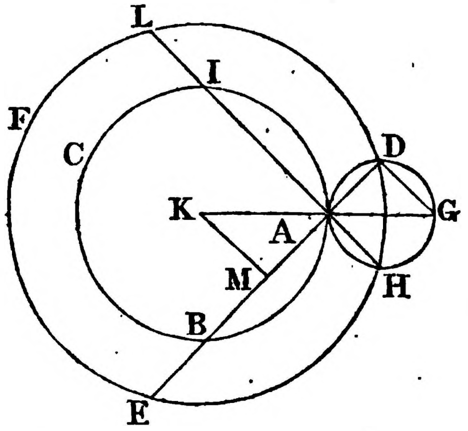
63. COR. If the diagonals of a quadrilateral rectilineal figure, inscribed in a circle, cut each other at right angles, the aggregate of the squares of the sides is the double of the square of the diameter of the circle.

For let the diagonals AB and CD of the quadrilateral figure ACBD, inscribed in the circle ACBD, cut one another at right $\angle\angle$ in E: Then it is evident from E. 47. 1, that $\overline{AC}^2+\overline{DB}^2$ is equal to the squares of the segments of the diagonals, that is, (S. 50. 3.) to the square of the diameter of the circle: Likewise $\overline{AD}^2+\overline{CB}^2$ may, in the same manner, be shewn to be equal to the square of the diameter; $\therefore$ $\overline{AC}^2+\overline{CB}^2+\overline{BD}^2+\overline{DA}^2=$ twice the square of the diameter of the circle.

PROP. LI.

64. PROBLEM. *To draw a straight line, cutting two concentric circles, so that the part of it which lies within the greater circle may be the double of the part which lies within the less.*

Let ABC, DEF be two given circles, having a common centre K: It is required to draw a straight line cutting ABC, DEF, so that the part of it within DEF shall be the double of the part of it within ABC.



Take any semidiameter as KA, of the circle ABC, and produce it to G, so that AG = AK; upon AG as a diameter describe the circle DAHG cutting DEF in D and H; join D, A, and H, A; and produce $\overline{DA}$ and $\overline{HA}$ to meet the circumferences again in B, E, and I, L, respectively: Then DE = 2AB; and HL=2AI.

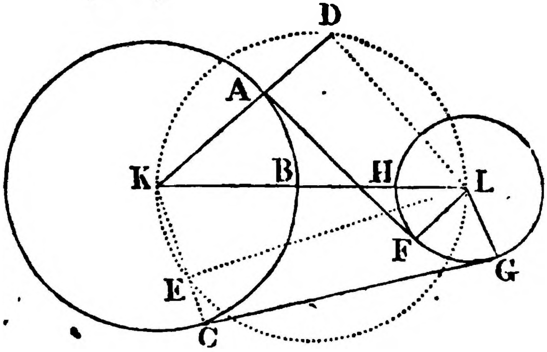
For join D, G and draw (E. 12. 1.) KM ⊥ to AB: And since (*constr.*) ADG is a semicircle, the ∠ ADG (E. 31. 3.) is a right ∠, as is (*constr.*) the ∠ KMA; also (E. 15. 1.) the ∠ KAM = ∠ DAG; and (*constr.*) the side KA, of the △ KMA, is equal to the side AG, of the △ ADG; ∴ (E.

26. 1.) AD=AM; ∴ MD=2MA, but (*constr.* and E. 3. 3.) DE=2MD and AB=2MA; ∴ DE=2AB: And, in the same manner, HL may be shewn to be the double of AI.

PROP. LII.

65. PROBLEM. *To draw a straight line which shall touch two given circles.*

Let ABC, HFG, be two given circles: It is required to draw a straight line which shall touch both the circles ABC, HFG.



First let the two circles be unequal. Find (E. 1. 3.) the centres K and L, of the two circles ABC, HFG; join K, L; upon KL as a diameter describe the circle DKC; from K as a centre at a distance =KB+LH, the aggregate of the semi-diameters of the two circles, describe a circle cutting $\overparen{KDL}$ in D, and draw $\overline{KD}$, cutting the circumference of ABC in A; ∴ KD=

$KB+LH$; in like manner, by the help of E. 3. 1, place, in the semi-circle KCL, $KE=KB\sim LH$, and let KE produced meet the circumference of ABC in C; from L draw (E. 31. 1.) LF parallel to DK, and LG parallel to KC; lastly, join A, F, and C, G: Then will $\overline{AF}$ and $\overline{CG}$, each of them, touch both the circles ABC, HFG.

For join D, L, and E, L: And since (*constr.*) $KE=KC\sim LG$, it is manifest that $EC=LG$; and (*constr.*) EC is parallel to LG; $\therefore$ (E. 33. 1.) CG is parallel to LE; but, since KEL is a semi-circle, the $\angle KEL$ is (E. 31. 3.) a right $\angle$; $\therefore$ (E. 29. 1.) the $\angle\angle$ KCG, CGL are right $\angle\angle$; $\therefore$ (E. 16. 3. *cor.*) CG touches both the circles ABC, HFG.

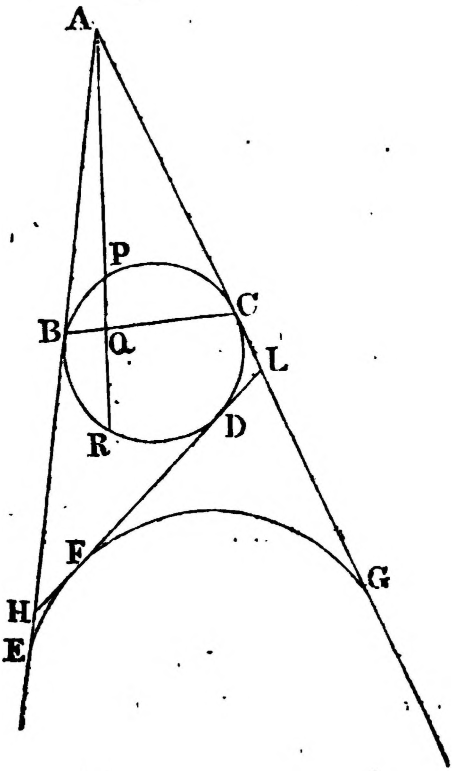
And, in the same manner, it may be shewn, that AF touches both the circles ABC, HFG.

Secondly, if the two given circles be equal to one another, a straight line may be drawn which shall touch them on contrary sides, in the same manner as when they are unequal: And, it is manifest (E. 33. 1. E. 29. 1. and E. 16. 3. *cor.*) that if a semi-diameter in each circle be drawn perpendicular to the straight line joining the two centres, the straight line, which joins the extremities of these two semi-diameters, will touch both the circles on the same side.

Prop. LIII.

66. Theorem. *If two straight lines, which touch two given circles, the one touching both the circles on the one side of them, the other on the other, be cut by a third tangent, which touches the two circles on contrary sides of them, then, of the segments into which the two first tangents are thus divided, those which are alternate are equal to one another.*

Let $\overline{ABE}$, $\overline{ACG}$ touch the two given circles



BCD, EFG, $\overline{ABE}$ on the one side of them, and $\overline{ACG}$ on the other; and let $\overline{HLFD}$ be drawn (S. 52. 3.) touching the two circles, on contrary sides of them: Of the segments into which $\overline{HL}$ divides $\overline{BE}$ and $\overline{CG}$, $\overline{BH}=\overline{LG}$, and $\overline{EH}=\overline{CL}$.

If the two circles be equal to one another, it is manifest, from the latter part of the demonstration of S. 52. 3., that $\overline{BE}$ and $\overline{CG}$ will be opposite sides of a ▭, and that, ∴, (E. 34. 1.) $\overline{BE}=\overline{CG}$: And, if $\overline{BE}$ be not parallel to $\overline{CG}$, but meets it, both the lines being produced, in A, then, since (S. 19. 3. *cor.* 1.) $\overline{AE}=\overline{AG}$ and $\overline{AB}=\overline{AC}$, ∴ $\overline{BE}=\overline{CG}$, as in the former case.

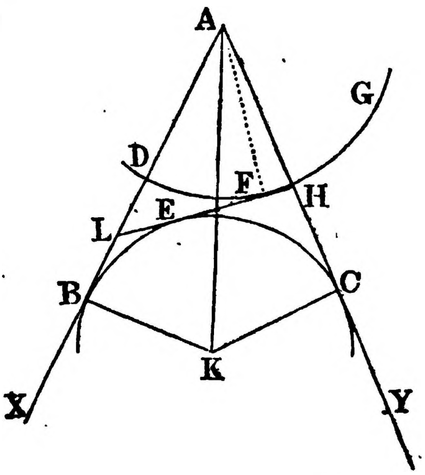
Again, (S. 19. 3. *cor.* 1.) $\overline{HB}+\overline{LC}=\overline{HD}+\overline{LD}$; or $\overline{HL}$; and $\overline{HL}=\overline{HF+FL}=\overline{HE}+\overline{LG}$; ∴ $\overline{HB}+\overline{LC}=\overline{HE}+\overline{LG}$; and, as hath been shewn, $\overline{BE}=\overline{CG}$; that is, $HB + HE = LC + LG$; if, ∴, these two equals be added to the equals $HB+LC$ and $HE+LG$, it is evident that $2\overline{HB}+\overline{HE}+\overline{LC}=2\overline{LG}+\overline{HE}+\overline{LC}$; take away from both $HE+LC$, and there remains $2\overline{HB}=2\overline{LG}$; ∴ $\overline{HB}=\overline{LG}$: And it has been proved that $\overline{BE}=\overline{CG}$; if, ∴, from these equals there be taken the equals $\overline{HB}$ and $\overline{LG}$, there will remain $\overline{EH}=\overline{CL}$.

PROP. LIV.

67. PROBLEM. *The perimeter, the vertical angle,*

and the altitude of a triangle being given, to construct the triangle.

Let XAY be a given rectilineal angle: It is



required to describe a triangle, which shall have XAY for its vertical angle, which shall have a given perimeter, and the perpendicular drawn from A to the opposite side, equal, also, to a given straight line.

From AX and AY cut off AB and AC, each of them equal to the half of the given perimeter; from B and C draw (E. 11. 1.) BK and CK ⊥ to AB and AC, respectively, and join A, K; ∴ (*constr.* and S. 73. 1.) KB = KC; from the centre K, at the distance KB, describe the circle BEC, which (*constr.* and E. 16. 3. *cor.*) will touch AB and AC in the points B and C; from AX cut off AD equal to the given perpendicular, and from the centre A at the distance AD describe the circle DFG.

Lastly, draw (S. 52. 3.) the straight line LH touching the circle BEC in E, and the circle DFG in F: Then is ALH the △ which was to be described.

For it has the ∠ XAY for its vertical ∠, and if A, F be joined, since $\overline{\text{LH}}$ touches the circle DFG in F, the ∠∠ AFL, AFH are (E. 18. 3.) right ∠∠; and (E. 15. def. 1.) $AF = AD$ which was made equal to the given perpendicular: Also (*constr.* and S. 19. 3. *cor.* 1.) $LE = LB$, and $HE = HC$; $\therefore LE + HE$, *i. e.* $LH = LB + HC$; to these equals add $AL + AH$; $\therefore AL + LH + HA = AL + LB + AH + HC$; but $AL + LB = AB$, and $AH + HC = AC$; $\therefore AL + LH + HA = AB + AC$; and AB and AC were made each of them equal to the half of the given perimeter; ∴ the △ ALH has its perimeter equal to the given perimeter; it has the given ∠ for its vertical ∠, and, as hath been shewn, it has its ⊥ AF equal to the given altitude.

PROP. LV.

68. THEOREM. *If the point, in which two straight lines that are perpendicular to each other meet, be applied to the circumference of a circle so that the straight lines themselves cut the circumference,*

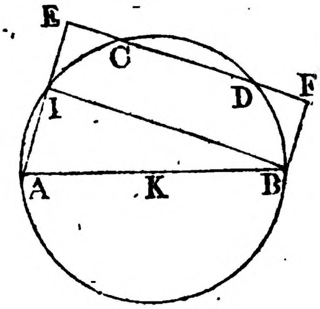
the centre of the circle is in the bisection of the straight line joining those two intersections.

For, the straight line, joining the intersections of the circumference and of the two straight lines which (*hyp.*) meet at some point of the circumference, and contain a right ∠, cuts off a semi-circle: If not, let it, if it be possible, cut off a segment greater than a semi-circle; ∴ (E. 31. 3.) the ∠ in that segment is less than a right ∠, which is contrary to the supposition: Neither can it cut off a segment less than a semi-circle; for, then the ∠ in that segment would be greater than a right ∠, which is, also, contrary to the supposition; ∴ the straight line joining the intersections cuts off a semi-circle, and ∴ passes through the centre of the circle, which point is ∴ in the bisection of that line.

Prop. LVI.

69. Theorem. *If from the extremities of any diameter, of a given circle, perpendiculars be drawn to any chord of the circle, that is not parallel to the diameter, the less perpendicular shall be equal to the segment of the greater contained between the circumference and the chord.*

From the extremities, A and B, of the diameter AB of the circle ABDC, let there be drawn AE



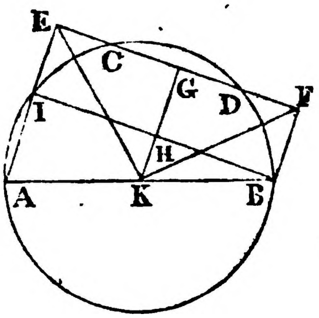
and BF ⊥ to the chord CD, which is not parallel to AB; let AE and BF meet CD, produced, in E and F; and let the greater ⊥ AE cut the circumference in I: Then BF = IE.

For join B, I: And since (*hyp.*) AICB is a semi-circle, ∴ (E. 31. 3.) the ∠ AIB, is a right ∠; and (*hyp.*) the ∠∠ IEF, EFD are, also, right ∠∠; ∴ (E. 28. 1.) the figure IEFB is a ▭; ∴ (E. 34. 1.) BF = IE.

PROP. LVII.

70. THEOREM. *If from the extremities of any diameter, of a given circle, perpendiculars be drawn to any chord of the circle, they shall meet the chord, produced, in two points which are equidistant from the centre.*

From the extremities, A, B, of the diameter AB, of the circle ABCD, of which K is the centre, let there be drawn AE, and BF, ⊥ to the chord



CD, and meeting CD produced in E and F, respectively: The points E and F are equidistant from K.

For, join K, E, and K, F, and B, I; and first let EF be parallel to AB: Then, since (*hyp.*) the ∠∠ AEF, EFD are right ∠∠, ∴ (E. 28. 1.) EA is, also, parallel to FB, and AEFB is a ▭; ∴ (E. 34. 1.) EA = FB, and the ∠ EAB = ∠ FBA, each of them being a right ∠; also the side KA, of the △ EAK, is equal to the side KB, of the △ FKB; ∴ (E. 4. 1.) EK = FK.

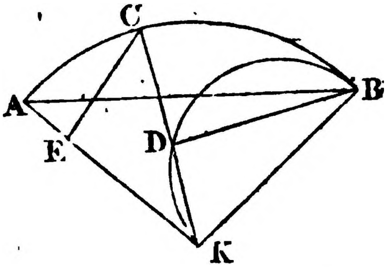
But, if EF be not parallel to AB, join B, I, and through K draw (E. 31. 1.) KHG parallel to AE or BF: And since (*hyp.*) AICD is a semi-circle, ∴ (E. 31. 3.) the ∠ AIB is a right ∠, as is, also, (*hyp.*) the ∠ AEF; ∴ (E. 28. 1.) EF is parallel to IB; and (*constr.*) the figures IG, HF, are ▭ ∴ (E. 34. 1.) EG = IH, and GF = HB; but since KG (*constr.*) is parallel to AE, and that the ∠∠ AIB, AEF are right ∠∠, ∴ (E. 29. 1.) the ∠∠ IHK, EGH are right ∠∠; and because KH, drawn from the centre K, cuts IB at right ∠∠, ∴ (E. 3. 3.) IH

= HB; ∴ EG = GF; and KG is common to the two $\triangle$s, KGE, KGF, and the $\angle$s KGE, KGF, as hath been shewn, are right $\angle$s; ∴ (E. 4. 1.) EK = FK.

PROP. LVIII.

71. THEOREM. *If upon either radius, bounding a quadrantal circular arch, as a diameter, a semi-circle be described, any chord of the semi-circle, drawn from the centre of the quadrant, shall be equal to the perpendicular distance of the point, in which the chord produced meets the quadrantal arch, from the other radius.*

Let BDK be a semi-circle, having for its diameter KB, one of the semi-diameters which bound the quadrantal circular arch $\overset{\frown}{ACB}$, and from the point C, in which any chord KD, of the semi-circle, meets, when produced, $\overset{\frown}{ACB}$, let CE be drawn perpendicular to KA the other ter-



minating semi-diameter of $\overset{\frown}{ACB}$: Then $KD = CE$.

For, since (*hyp.*) KDB is a semi-circle, ∴ (E. 31. 3.) the $\angle BDK$ is a right $\angle$; as is, also, (*hyp.*) the $\angle CEK$: Again, since (*hyp.*) $\overset{\frown}{ACB}$ is a quadrant of the circumference of its circle, ∴ (E. 27. 3. and E. 15. 1. *cor.* 2.) the $\angle AKB$ is a right $\angle$; ∴ the $\angle EKC + \angle DKB =$ a right $\angle$; also, since the $\angle CEK$ is a right $\angle$, ∴ (E. 32. 1.) the $\angle EKC + \angle KCE =$ a right $\angle$; ∴ the $\angle EKC + \angle DKB = \angle EKC + \angle KCE$; ∴ the $\angle DKB = \angle KCE$; and the side KB, of the $\triangle KDB$, is (E. 15. def. 1.) equal to the side KC, of the $\triangle CEK$; ∴ (E. 26. 1.) $KD = CE$.

PROP. LIX.

72. THEOREM. *If the angle contained by two straight lines, one of which cuts a circle and the other meets it, be equal to the angle in the alternate segment of the circle, the straight line which meets, shall touch the circle.*

For if the straight line which, in this case, meets the circle, does not touch it, from the point in which it meets the circle, draw (E. 17. 3.) a straight line touching the circle; Then (*hyp.* and E. 32. 3.) it is manifest that the greater of two angles is equal to the less; which is absurd.

Prop. LX.

73. Theorem. *A straight line touching a circular arch in the bisection of that arch, is parallel to its chord.*

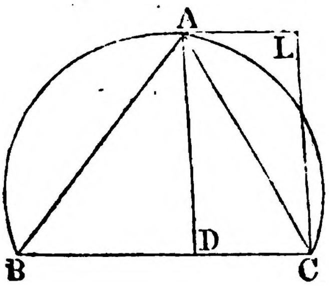
For the ∠∠, which each half of the arch subtends at the opposite extremity of the chord, are (E. 27. 3.) equal to one another; and (E. 32. 3.) they are also equal to the ∠∠ which the straight lines, joining the bisection of the arch and the extremities of the chord, make with the straight line that touches the arch at its bisection; ∴ (E. 27. 1.) the tangent, at that point of bisection, is parallel to the chord.

Prop. LXI.

74. Problem. *The base, the vertical angle, and the altitude of a triangle being given, to construct the triangle.*

Let BC be the given base of a △, of which the vertical ∠, and the altitude are also given: It is required to construct the triangle.

Upon BC describe (E. 33. 3.) a segment of a circle BAC, capable of containing an ∠ equal to the given vertical ∠; from C draw (E. 11. 1.) CL ⊥ to BC, and make it equal to the given alti-



tude of the △; through L draw $\overline{LA}$ parallel to $\overline{BC}$, and let $\overline{LA}$ meet $\overset{\frown}{BAC}$ in A; join B, A and C, A: Then is ABC the △ which was to be constructed.

For draw (E. 31. 1.) AD parallel to LC; ∴, the figure ADCL is a ▭, and (E. 34. 1.) AD = LC: And because AD is parallel to LC, and (*constr.*) the ∠LCD is a right ∠, ∴ (E. 29. 1.) the ∠ADC is a right ∠; *i. e.* $\overline{AD}$ is ⊥ to BC, and it has been shewn to be equal to LC, which (*constr.*) is equal to the given perpendicular. Also (*constr.*) the ∠BAC is equal to the given vertical ∠; ∴ ABC is the △ which was to be constructed.*

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* If the straight line LA drawn from the extremity of CL, which is made equal to the given perpendicular, fall without the segment BAC, the problem is manifestly impossible: If LA touch the circle BAC, the problem has only one solution; but if LA cut the segment BAC, the problem admits of two solutions.

PROP. LXII.

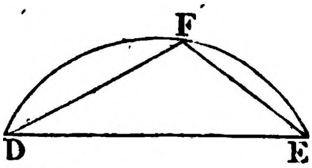
75. PROBLEM. *To find a point in a given straight line, from which if straight lines be drawn to two given points, on the same side of the given line, they shall contain an angle equal to a given rectilineal angle.*

Upon the straight line joining the two given points, describe (E. 33. 3.) a segment of a circle, capable of containing an ∠ equal to the given rectilineal ∠, and the point in which it meets, or cuts, the given straight line, is evidently the point which was to be found: And if the circumference of the segment, so described, cut the given straight line, it is manifest that the problem admits of two solutions: But if the circumference of the segment neither touch nor cut the given line, the problem is impossible.

PROP. LXIII.

76. PROBLEM. *The vertical angle, the base, and the aggregate of the three sides of a triangle being given, to construct the triangle.*

Let DE be the given base: It is required to describe on DE a △, which shall have its two remaining sides equal together, to a given finite



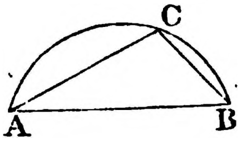
straight line, and the ∠ contained by them equal to a given rectilineal angle.

Upon DE describe (E. 33. 3.) a segment of a circle DFE, capable of containing an ∠ equal to the given rectilineal ∠; divide (S. 47. 3.) $\overset{\frown}{BFE}$, in F, so that the aggregate of the chords of $\overset{\frown}{DF}$, $\overset{\frown}{EF}$ shall be equal to the aggregate of the two remaining sides of the Δ; and join D, F and E, F: Then it is manifest that DFE is the Δ which was to be constructed.

Prop. LXIV.

77. Problem. *The vertical angle, the base, and the excess of the greater of the two remaining sides, of a scalene triangle, above the less, being given, to construct the triangle.*

Let AB be the given base: It is required to describe on AB a Δ, which shall have the difference of its two remaining sides equal to a given finite straight line, and the ∠ contained by them equal to a given rectilineal angle.

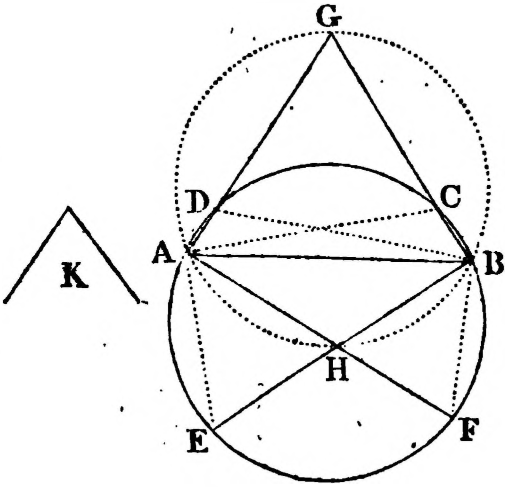


Upon AB describe (E. 33. 3.) a segment of a circle capable of containing an ∠ equal to the given rectilineal ∠; divide (S. 48. 3.) $\overset{\frown}{ACB}$ in C, so that the difference of the chords of $\overset{\frown}{AC}$ and $\overset{\frown}{BC}$ may be equal to the excess of the greater of the two remaining sides of the △ above the less; and join A, C and B, C: Then it is evident that ACB is the △ which was to be constructed.

PROP. LXV.

78. PROBLEM. *From two given points, in the circumference of a circle, to draw two equal chords of that circle, which, produced if necessary, shall make with one another an angle equal to a given rectilineal angle.*

Let A and B be two given points in the circumference of the circle AEFB; and let K be a given rectilineal angle: It is required to draw, from A and B, two equal chords of the circle AEFB, which make with one another an ∠ equal to the ∠K.



Join A, B; upon $\overline{AB}$ describe (E. 33. 3.) a segment of a circle AGB capable of containing an ∠ = the ∠ K, and complete the circle AGBH; bisect (E. 30. 3.) $\widehat{AGB}$ in G, and $\widehat{AHB}$ in H; draw $\overline{AG}$ and $\overline{BG}$, cutting $\widehat{ADB}$ in D and C; also draw $\overline{AH}$ and $\overline{BH}$, and produce them to meet $\widehat{AEB}$ in F and E.

It is manifest, from the construction, that the ∠ AGB, which the two chords, AD, BC, make with one another when produced, = ∠ K; also, since (E. 22. 3.) the ∠ AGB + ∠ AHB = two right ∠∠, and that (E. 13. 1.) the ∠ AHE + ∠ AHB = two right ∠∠, ∴ the ∠ AHE = ∠ AGB; but (*constr.*) the ∠ AGB = ∠ K; ∴ the ∠ AHE, which the two chords AF, BE, make with one another, is equal to the ∠ K.

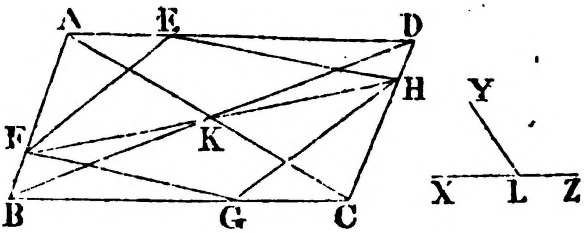
Again, join A, C, and B, D; and since (*constr.*) $\overset{\frown}{AG} = \overset{\frown}{BG}$ ∴ (E. 29. 3.) $\overline{AG} = \overline{BG}$; ∴ (E. 5. 1.) the $\angle GAB = \angle GBA$; also (E. 21. 3.) the $\angle ADB = \angle ACB$; ∴ (S. 26. 1.) the third $\angle ABD$, of the $\triangle ADB$, is equal to the third $\angle BAC$ of the $\triangle BCA$; ∴ (E. 26. 3.) $\overset{\frown}{AD} = \overset{\frown}{BC}$, and (E. 29. 3.) the chord $AD =$ the chord BC.

Lastly, if A, E, and B, F, be joined, it may be shewn, in the same manner, that the $\angle HAB = \angle HBA$, that (E. 21. 3.) the $\angle EAF = \angle FBE$, and ∴ that the $\angle EAB = \angle FBA$; ∴ (E. 26. 3. and E. 29. 3.) $\overline{BE} = \overline{AF}$.

PROP. LXVI.

79. PROBLEM. *In a given parallelogram to inscribe a parallelogram which shall have one of its angles equal to a given angle, and posited in a given point of one of the sides of the given parallelogram.*

Let F be a given point in the side AB of the



▭ ABCD; and YLX a given rectilineal angle: It is required to inscribe, in the ▭ ABC, a ▭ which shall have one of its ∠∠ posited in F, and equal to the ∠ YLX.

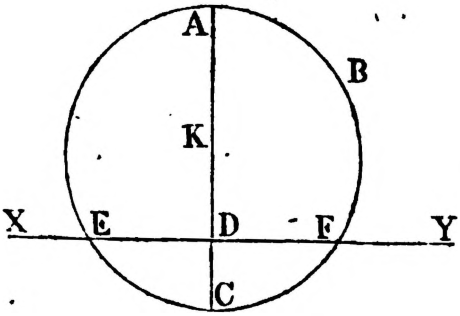
Join B, D; bisect $\overline{\text{BD}}$ (E. 10. 1.) in K; draw $\overline{\text{FK}}$ and produce it to meet DC in H; ∴ (*constr.* E. 15. 1. E. 29. 1. and E. 26. 1.) DH = BF; produce XL to Z; upon FH describe (E. 33. 3.) a segment of a circle capable of containing an ∠ = ∠ YLZ, and let its circumference cut AD in E; if ∴ F, E and H, E be joined, it is plain that the ∠ FEH = ∠ YLZ; from CB cut off CG = AE; join F, G and H, G: Then is FEHG the figure which was to be described.

For (*constr.* and S. 43. 1.) EFGH is a ▭; ∴ (E. 29. 1.) the ∠ HEF + ∠ EFG = two right ∠∠; also (E. 3. 1.) the ∠ YLZ + ∠ YLX = two right ∠∠; and it has been shewn, that the ∠ HEF = ∠ YLZ; ∴ the ∠ EFG = ∠ YLX; and it is posited in the given point F. Therefore, &c.

PROP. LXVII.

80. PROBLEM. *To produce a given straight line so that the rectangle, under the given straight line, and the part of it produced, shall be equal to a given square.*

Let AD be a given finite straight line: It is



required to produce AD, so that the rectangle contained by AD and the part produced may be equal to a given square.

Through D draw any straight line XY, and from DX and DY cut off (E. 3. 1.) DE and DF each of them equal to the side of the given square; describe (S. 5. 1. *cor.*) a circle which shall pass through the three points A, E, and F; and produce AD to meet the circumference in C: Then (E. 35. 3.) it is manifest that the rectangle $\overline{AD}\times\overline{DC} = \overline{ED}\times\overline{DF}$ or $\overline{ED}^2$; but $\overline{ED}$ was made equal to the side of the given square; ∴ $\overline{AD}$ has been produced to C, so that $\overline{AD}\times\overline{DC}$ is equal to the given square.

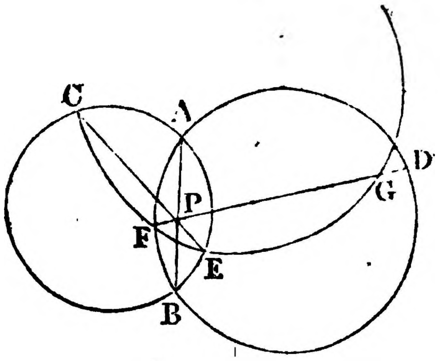
81. Cor. By a similar method, it is manifest, a given straight line may be produced, so that the rectangle contained by the straight line, and the part produced shall be equal to a given rectangle: That is, if three straight lines be given, a fourth may be found so that the rectangle, contained by it and any of the three given straight lines, shall

be equal to the rectangle contained by the remaining two.

PROP. LXVIII.

82. THEOREM. *If through any point in the common chord of two circles, which intersect one another, there be drawn any two other chords, one in each circle, their four extremities shall all lie in the circumference of a circle.*

Let P be any point in $\overline{AB}$, which is a common chord of the two circles ABC, ABD; and through P let there be drawn a chord CPE, of the circle ABC, and FPD a chord of the circle ABD; the four points C, F, E, D, lie in the circumference of a circle.



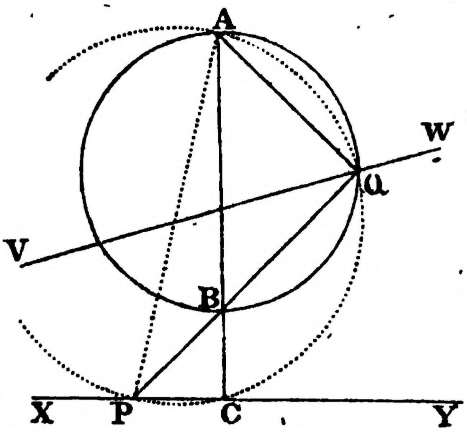
For describe (S. 5. 1. *cor.*) a circle CFE, which shall pass through the three points C, F and E; it shall, also, pass through D: If not let it cut FD in some other point as G.

Then (E. 35. 3.) $\overline{CP} \times \overline{PE} = \overline{AP} \times \overline{PB}$; and $\overline{AP} \times \overline{PB} = \overline{FP} \times \overline{PD}$; $\therefore \overline{CP} \times \overline{PE} = \overline{FP} \times \overline{PD}$; but (E. 35. 3.) $\overline{CP} \times \overline{PE} = \overline{FP} \times \overline{PG}$; $\therefore \overline{FP} \times \overline{PD} = \overline{FP} \times \overline{PG}$; *i. e.* the greater rectangle is equal to the less, which is absurd; therefore the circle which passes through C, F, E cannot pass otherwise than through D.

Prop. LXIX.

83. Theorem. *If through the given extremity of any diameter of a circle straight lines be drawn to meet an indefinite straight line without the circle, which is perpendicular to the diameter produced, the rectangles contained by the segments of these lines lying between the given point, the point in which each of them cuts the circumference again, and the indefinite line, shall be equal to each other.*

Through the extremity B of the diameter AB, of the circle AQB, let there be drawn any number of straight lines, terminated one way by the circumference, and the other way by the indefinite straight line XY, which meets AB, produced, at right angles in C: The rectangles contained by the segments into which the lines so drawn are divided by the point B, shall be equal to one another.



For, let $\overline{PBQ}$ be any of the lines so drawn through B; join A, P and A, Q: And because (*hyp.*) AQB is a semi-circle, ∴ (E. 31. 3.) the ∠ AQP is a right ∠, as is, also, (*hyp.*) the ∠ ACP; ∴ (S. 29. 1. *cor.* 2.) a circle described upon AP as a diameter, will pass through Q and C; ∴ (E. 35. 3.) $\overline{PB} \times \overline{BQ} = \overline{AB} \times \overline{BC}$; and, in the same manner, it may be shewn that the rectangle contained by the segments of any other straight line, so drawn through B, is equal to $\overline{AB} \times \overline{BC}$, and, ∴, equal also to $\overline{PB} \times \overline{BQ}$. All such rectangles are, ∴, equal to one another.

84. Cor. Hence, through a given point, (B) between an indefinite straight line (XY) and a line of any kind (VW), in the same plane with it, a straight line may be drawn to meet the two given lines, so that the rectangle, contained by the segments into which it is divided by the given point, shall be equal to a given square.

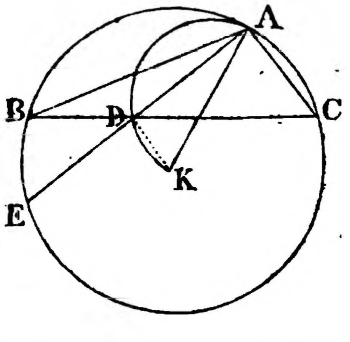
For draw (E. 12. 1.) BC ⊥ to XY, and produce

CB (S. 67. 3.) to A, so that $\overline{CB}\times\overline{BA}$ may be equal to the given square; upon AB, as a diameter, describe the circle ABQ,* cutting VW in Q; lastly, draw QB, and produce it to meet XY in P: Then since (S. 69. 3.) $\overline{PB}\times\overline{BQ}=\overline{CB}\times\overline{BA}$, and that (*constr.*) $\overline{CB}\times\overline{BA}$ is equal to the given square; ∴ $\overline{PB}\times\overline{BQ}$ is, also, equal to the given square.

PROP. LXX.

85. PROBLEM. *From the obtuse angle of an obtuse-angled triangle, to draw a straight line to the base, the square of which shall be equal to the rectangle contained by the segments, into which it divides the base.*

Let BAC be an obtuse-angled Δ, obtuse-



angled at A: It is required to draw from A to

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* If the circumference of the circle ABQ do not cut VW, the problem admits not of a solution.

BC a straight line, the square of which shall be equal to the rectangle of the segments into which it divides BC.

About BAC describe (S. 5. 1. *cor.*) a circle ABEC, and take its centre K; join K, A; upon KA, as a diameter, describe the circle ADK cutting BC in D; join A, D: Then $\overline{AD}^2 = \overline{BD} \times \overline{DC}$.

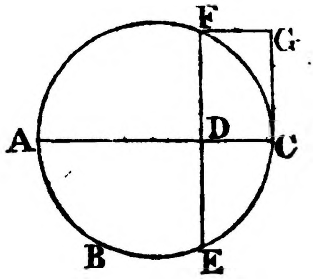
For produce AD to meet the circumference in E, and join K, D: And since (*constr.*) ADK is a semi-circle; ∴ (E. 31. 3.) the ∠ ADK is a right ∠; ∴ (E. 3. 3.) $\overline{AD} = \overline{DE}$; but (E. 35. 3.) $\overline{BD} \times \overline{DC} = \overline{AD} \times \overline{DE}$; ∴ $\overline{BD} \times \overline{DC} = \overline{AD}^2$.

86. Cor. A segment of a circle being given, that is less than a semi-circle, the method of drawing, from any point of its circumference, a chord of the circle, that shall be bisected by the chord of the segment, is shewn in the solution of the above problem.

Prop. LXXI.

87. Problem. *To make a rectangle which shall be equal to a given square, and shall have its two adjacent sides, together, equal to a given straight line; the side of the given square being less than the half of the given straight line.*

Let AC be a given straight line: It is required to make a rectangle, which shall be equal to a



given square, and which shall have its two adjacent sides, together, equal to $\overline{AC}$.

Upon $\overline{AC}$, as a diameter, describe the circle ABC; from C draw (E. 11. 1.) $\overline{CG} \perp$ to $\overline{AC}$, and make $\overline{CG}$ equal to the side of the given square; through G draw (E. 31. 1.) $\overline{GF}$ parallel to $\overline{AC}$, and through F draw $\overline{FDE}$ parallel to $\overline{CG}$: The rectangle $\overline{AD} \times \overline{DC}$ is equal to the given square.

For (*constr.*) DCGF is a ▭; ∴ (E. 34. 1.) DF $=$ CG, and ∴ (*constr.*) DF $=$ the side of the given square: Again, because (*constr.*) DF is parallel to CG, and the ∠ ACG is a right ∠, ∴ (E. 29. 1.) the ∠ CDF is, also, a right ∠; and (*constr.*) $\overline{ADC}$ is the diameter of the circle ABC; ∴ (E. 3. 3.) DF $=$ DE; but (E. 35. 3.) $\overline{AD} \times \overline{DC} = \overline{DF} \times \overline{DE}$; *i. e.*, since DF $=$ DE, $\overline{AD} \times \overline{DC} = \overline{DF}^2$ or $\overline{CG}^2$; ∴ $\overline{AD} \times \overline{DC}$ is equal to the given square, and AD together with DC make up the given straight line AC.

88. Cor. 1. If the side of the given square be greater than the half of the given straight line, the problem admits of no solution.

89. Cor. 2. In the same manner, the greater side of a given oblong may be divided into two parts, so that the rectangle contained by them shall be equal to the given oblong, a square having first (E. 14. 2.) been found that is equal to the oblong: But, in this case, the half the greater side of the oblong must exceed the double of the lesser side.

90. Cor. 3. In the same manner, also, a straight line may be divided into two parts, so that the rectangle contained by them, shall be equal to a given rectangle; if the side of a square which is equal to the given rectangle, do not exceed the half of the given straight line.

91. Cor. 4. If the measure of the surface of an oblong be given, and if its perimeter be also given, the rectangle itself may hence be constructed.

Prop. LXXII.

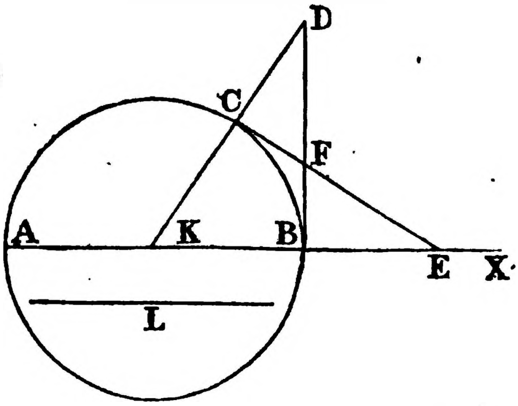
92. Theorem. *If from a given point without a circle, two equal straight lines be drawn to the convex circumference, one of which touches the circle, the other shall also touch it.*

For, if not, draw (E. 17. 3.) from the given point a straight line to touch the circle, and (S. 19. 3. *cor.* 1.) it will be equal to the other tangent; and thus more than two equal straight lines can be drawn from a point, without a circle, to the circumference; which (E. 8. 3.) is absurd; ∴ the straight line which is drawn from the same point, without the circle, as the tangent, and which is equal to the tangent, itself also touches the circle.

Prop. LXXIII.

93. **Problem.** *To produce a given straight line, so that the rectangle contained by the whole line thus produced, and the part of it produced, shall be equal to a given square.*

Let AB be a given straight line, and L the side



of a given square: It is required to produce AB so that the rectangle, contained under the whole line produced, and the part of it produced, may be equal to the square of L.

Bisect (E. 10. 1.) AB in K, and upon AB, as a diameter, describe the circle ABC; from B draw (E. 11. 1.) $\overline{BD} \perp$ to $\overline{AB}$, and make $\overline{BD} = L$; join K, D, and let $\overline{KD}$ cut the circumference in C; from C draw $\overline{CE} \perp$ to $\overline{KC}$, and let $\overline{CE}$ meet $\overline{AB}$ produced in E. Then since (*constr.* and E. 16. 3. *cor.*) $\overline{CE}$ touches the circle, and EBA cuts it, ∴ (E. 36. 3.) $\overline{AE} \times \overline{EB} = \overline{EC}^2$; but, since the $\angle\angle$ KBD, KCE are right $\angle\angle$, and the $\angle$ at K is common to the two $\triangle\triangle$ KBD, KCE, and that (E. 15. def. 1.) the side KD = the side KC, ∴ (E. 26. 1.) $\overline{EC} = \overline{BD}$; but (*constr.*) $\overline{BD} = L$; and it has been shewn that $\overline{AE} \times \overline{EB} = \overline{EC}^2$; ∴ $\overline{AE} \times \overline{EB} =$ the square of L.

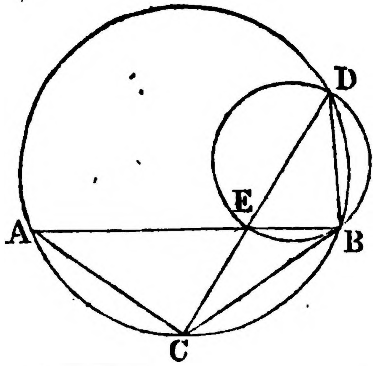
94. Cor. By the help of this proposition and (E. 14. 2.), a given straight line may be produced, so that the rectangle contained by the whole line thus produced, and the part of it produced, shall be equal to a given rectilineal figure.

Prop. LXXIV.

95. Theorem. *If, from the bisection of any given arch of a circle, a straight line be drawn cutting the chord of that arch, or the chord produced, and the circumference also of the circle, the rectangle*

contained by the two parts of the straight line so drawn, the one lying between the point of bisection and the circumference, the other between the point of bisection and the chord, shall be equal to the square of the chord, of half the arch.

Let AB be the chord and let C be the bisection, of the arch $\widehat{ACB}$ of the circle ADBC ; and, first, let any straight line CD be drawn cutting the chord in E, and then meeting the circumference of the circle in D ; also let there be drawn $\overline{CB}$, the chord of $\widehat{CB}$, the half of $\widehat{ACB}$: Then $\overline{DC} \times \overline{CE} = \overline{CB}^2$.

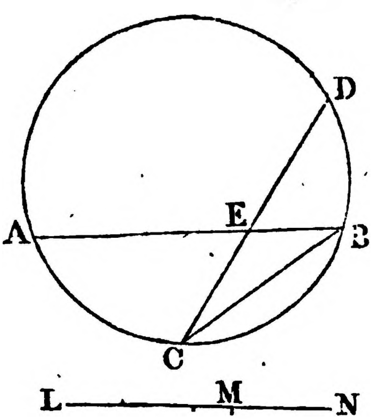


For join C, A and B, D, and about the $\triangle$DBE describe (S. 5. 1. *cor.*) the circle DEB: And because (*hyp.*) $\widehat{AC} = \widehat{CB}$, ∴ (E. 27. 3.) the $\angle ABC = \angle CAB$; and (E. 21. 3.) the $\angle CAB = \angle CDB$; ∴ the $\angle ABC = \angle BDE$; ∴ (S. 59. 3.) the straight line CB touches the circle DEB in B ; ∴ (E. 36. 3.) $\overline{DC} \times \overline{CE} = \overline{CB}^2$.

Prop. LXXV.

96. **Problem.** *From the bisection of a given arch of a circle, to draw a straight line, such that the part of it intercepted between the chord, or the chord produced, of the given arch and the circumference, shall be equal to a given straight line.*

Let $\widehat{ACB}$ be a given arch of the circle ADBC;



let $\overline{AB}$ be its chord, and C its bisection, and let LM be a given straight line: It is required to draw from C a straight line such that the part of it between AB, and the circumference ADB shall be equal to LM.

Join C, B; and produce (S. 73. 3.) LM to N, so that $\overline{LN}\times\overline{NM}=\overline{CB}^2$; from C as a centre, at a distance equal to LN, describe a circle cutting

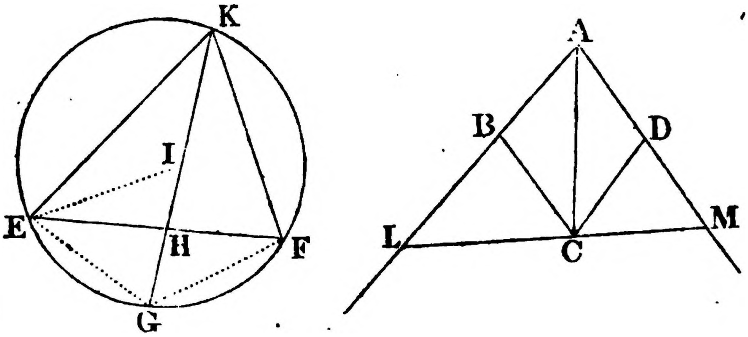
$\overset{\frown}{ADB}$ in D; join C, D, and let $\overline{CD}$ cut $\overline{AB}$ in E: Then is $\overline{ED} = \overline{LM}$.

For (S. 74. 3.) $\overline{DC} \times \overline{CE} = \overline{CB}^2$; and (*constr.*) $\overline{LN} \times \overline{NM} = \overline{CB}^2$; $\therefore$ $\overline{DC} \times \overline{CE} = \overline{LN} \times \overline{NM}$; but (*constr.*) $\overline{DC} = \overline{LN}$; $\therefore$ CE = NM; and $\therefore$ $\overline{ED} = \overline{LM}$.

Prop. LXXVI.

97. Problem. *Through any given angle of a given equilateral four-sided figure, to draw a straight line terminated by the sides produced, containing the angle opposite to the given angle, which shall be equal to a given straight line.*

Let ABCD be a given equilateral rhombus, and



EF a given straight line: Through any of the angular points of ABCD, as C, it is required to draw a straight line, terminated by AB and AD produced, which shall be equal to $\overline{EF}$.

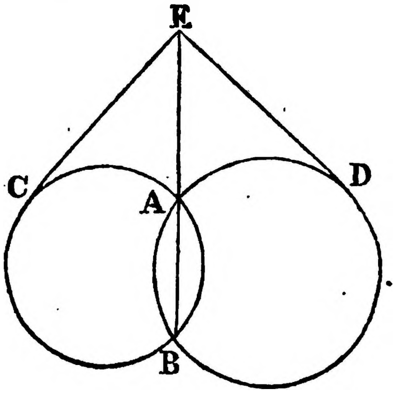
Join A, C; upon EF describe (E. 33. 3.) a segment of a circle EKF, capable of containing an $\angle$ equal to the $\angle$ BAD of the rhombus, and complete the circle; bisect (E. 30. 3.) $\overset{\frown}{EGF}$ in G; from G draw (S. 75. 3.) GHK so that $HK = AC$; join E, K; in AB produced, take (E. 3. 1.) $AL = KE$; join L, C, and produce LC to meet AD produced in M: Then $\overline{LCM} = \overline{EF}$.

For join K, F, and G, E, and G, F: And because (E. 32. def. 1.) BA, AC are equal to DA, AC, each to each, and that the base BC, of the $\triangle$ ABC, is equal to the base DC, of the $\triangle$ ADC, $\therefore$ (E. 8. 1.) the $\angle BAC = \angle DAC$, and the $\angle$ BAC is, $\therefore$, the half of the $\angle$ BAD: Again, because (*constr.*) $\overset{\frown}{EG} = \overset{\frown}{FG}$, $\therefore$ (E. 27. 3.) the $\angle EKG = \angle FKG$, and the $\angle$ EKG is, $\therefore$, the half of the $\angle$ EKF, which (*constr.*) is equal to the $\angle$ BAD; $\therefore$ the $\angle EKH = \angle LAC$; and (*constr.*) the two sides EK, KH of the $\triangle$ EKH, are equal to the two sides LA, AC, of the $\triangle$ LAC; $\therefore$ (E. 4. 1.) $LC = EH$, and the $\angle ACL = \angle KHE$; $\therefore$ (E. 13. 1.) the $\angle KHF = \angle ACM$; also, as hath been shewn, the $\angle CAM = \angle HKF$, and the side KH (*constr.*) of the $\triangle$ KHF is equal to the side AC of the $\triangle$ ACM; $\therefore$ (E. 26. 1.) $CM = HF$; and it has been proved that $LC = EH$; $\therefore LC + CM = EH + HF$; that is, $LM = EF$.

PROP. LXXVII.

98. THEOREM. *If two circles cut each other, and from any point, in the straight line produced, which joins their intersections, two tangents be drawn, one to each circle, they shall be equal to one another.*

Let the two circles ACB, ADB, cut one another



in the points A and B, and from any point E in AB, produced, let there be drawn $\overline{EC}$ and $\overline{ED}$ touching the circles ACB, ADB, in the points C and D respectively: $\overline{EC} = \overline{ED}$.

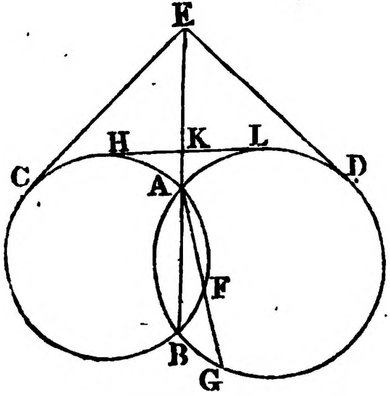
For (E. 36. 3.) $\overline{EC}^2 = \overline{BE} \times \overline{EA}$; also $\overline{ED}^2 = \overline{BE} \times \overline{EA}$; $\therefore \overline{EC}^2 = \overline{ED}^2$; and $\therefore \overline{EC} = \overline{ED}$.

99. COR. The straight line AB which passes through the intersections of two circles ACB, ADB, that cut one another, bisects the straight line HL, which touches both the circles.

PROP. LXXVIII.

100. THEOREM. *If two circles cut each other, and if two tangents drawn, one to each circle, from any point without them, be equal, the straight line, joining the intersections of the circles, shall, if it be produced, pass through the common extremity of the equal tangents.*

Let the two circles ACB, ADB, cut one another in A and B, and from any point E, without the circles, let there be drawn $\overline{EC}$ touching the circle ACB, and $\overline{ED}$ touching the circle ADB: If $\overline{EC} = \overline{ED}$, the points E, A, and B, are in the same straight line.



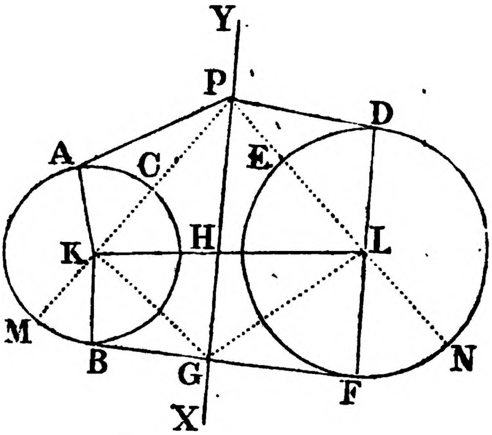
For join E, A; then shall $\overline{EA}$ produced pass through B; if not, let it pass otherwise, as $\overline{EAFG}$: Then (E. 36. 3.) $\overline{FE} \times \overline{EA} = \overline{EC}^2$; also $\overline{GE} \times \overline{EA}$

$=\overline{ED}^2$; and (*hyp.*) $\overline{EC}^2=\overline{ED}^2$; $\therefore\ \overline{FE}\times\overline{EA}=\overline{GE}\times\overline{EA}$; $\therefore\ \overline{FE}=\overline{GE}$; *i. e.* the less of two straight lines is equal to the greater, which is impossible; $\therefore\ \overline{EA}$, produced, cannot pass otherwise than through B; so that the three points E, A, and B are in the same straight line.

PROP. LXXIX.

101. PROBLEM. *Two circles being given, neither of which lies within the other, to draw a straight line, such that the tangents to the two circles, drawn from any point of the line, shall be equal to one another.*

Let ABC, DEF, be two given circles, neither



of which lies within the other: It is required to draw a straight line, such that the tangents to the two circles, drawn from any point of the line, shall be equal to one another.

Find (E. 1. 3.) the centres K and L, of the two given circles, and draw $\overline{KL}$; draw (S. 52. 3.) $\overline{BF}$ touching the two circles, on the same side, in B and F; bisect (E. 18. 1.) BF in G; and through draw (E. 12. 1.) $\overline{XY} \perp$ to $\overline{KL}$: The tangents drawn to the two circles ABC, DEF, from any point in $\overline{XY}$ are equal to one another.

For take any point P in $\overline{XY}$, and draw (E. 17. 3.) from P the straight lines $\overline{PA}$ and $\overline{PD}$, touching the circles in A and D respectively; and draw $\overline{PK}$, $\overline{AK}$, $\overline{BK}$, $\overline{KG}$, $\overline{LG}$, $\overline{LD}$ and $\overline{LP}$.

And because (*constr.*) the $\angle\angle$ at H are right $\angle\angle$; $\therefore$ (E. 47. 1.) $\overline{PK}^2+\overline{LG}^2=\overline{PH}^2+\overline{HK}^2+\overline{LH}^2+\overline{HG}^2$; and $\overline{PL}^2+\overline{KG}^2=$ the same four squares; $\therefore \overline{PK}^2+\overline{LG}^2=\overline{PL}^2+\overline{KG}^2$; but (*constr.* and E. 18. 3.) the $\angle\angle$ PAK, KBG, GFL, and LDP, are right $\angle\angle$; $\therefore$ (E. 47. 1.) $\overline{PK}^2+\overline{LG}^2=\overline{PA}^2+\overline{AK}^2+\overline{LF}^2+\overline{FG}^2$; and $\overline{PL}^2+\overline{KG}^2=\overline{PD}^2+\overline{LF}^2+\overline{AK}^2+\overline{GF}^2$, because (*constr.*) $\overline{GF}=\overline{GB}$; and $\overline{DL}=\overline{LF}$, and $\overline{KB}=\overline{KA}$; $\therefore \overline{PA}^2+\overline{AK}^2+\overline{LF}^2+\overline{FG}^2=\overline{PD}^2+\overline{LF}^2+\overline{AK}^2+\overline{FG}^2$; take away, $\therefore$, from both, the squares of AK, of $\overline{LF}$ and of $\overline{FG}$, and there remains $\overline{PA}^2=\overline{PD}^2$; $\therefore \overline{PA}=\overline{PD}$.

102. Cor. 1. The difference of the squares of the distances of any point P in the line XY so drawn, from the centres K and L, of the two given

circles, is equal to the difference of the squares of the two semi-diameters of the circles.

103. Cor. 2. If from any point P in the straight line XY, so drawn, any two straight lines be drawn, PCM, PEN, the one of them cutting the one of the given circles, the other the other, the rectangles, $\overline{MP} \times \overline{PC}$, $\overline{NP} \times \overline{PE}$, contained by the whole lines and the parts of them without the circles, shall be equal to one another.

For draw (E. 17. 3.) from P, $\overline{PA}$ touching the circle ABC, and $\overline{PD}$ touching the circle ADE: Then (E. 36. 3.) $\overline{MP} \times \overline{PC} = \overline{PA}^2$; and $\overline{NP} \times \overline{PE}$ $\overline{PD}^2$; but (S. 79. 3.) $\overline{PA}^2 = \overline{PD}^2$; $\therefore$ $\overline{MP} \times \overline{PC} = \overline{NP} \times \overline{PE}$.

Prop. LXXX.

104. Problem. *To find a point from which if straight lines be drawn to touch three given circles, none of which lies within another, the tangents so drawn shall be equal to one another.*

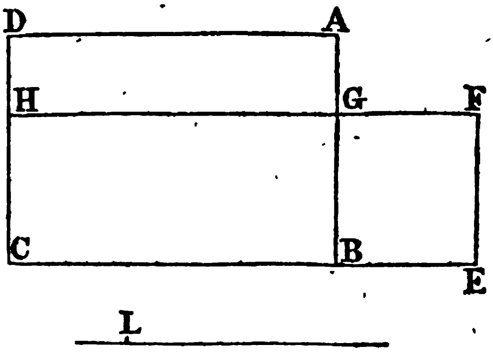
Draw (S. 79. 3.) the straight line which is the *locus* of equal tangents drawn to two of these given circles; draw likewise, the straight line, which is the *locus* of equal tangents drawn to the remaining circle and to either of the two circles first taken: It is manifest that the intersection of the two straight lines, so drawn, will be the point which was to be found.

105. Cor. If from the point, thus found, any number of straight lines be drawn cutting the three given circles, the rectangles contained by the whole lines, so drawn, and the parts of them without the circles, shall (E. 36. 3. and S. 80. 3.) be equal to one another.

Prop. LXXXI.

106. Problem. *To divide a given straight line into two parts, so that the square of the one shall be equal to the rectangle contained by the other and a given straight line.*

Let AB and L be two given finite straight



lines: It is required to divide $\overline{AB}$ into two parts, so that the square of the one shall be equal to the rectangle contained by the other and by the given line L.

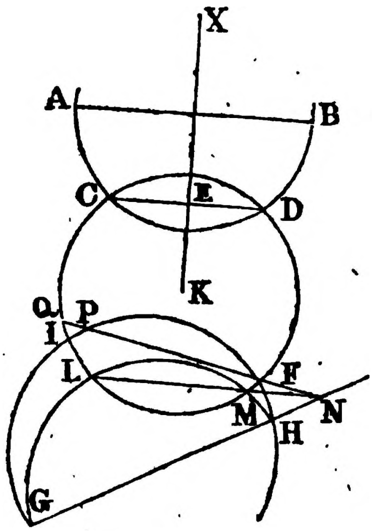
From B draw (E. 11. 1.) $\overline{BC} \perp$ to $\overline{AB}$; make $\overline{BC} = L$, and (E. 31. 1.) complete the ▭ ABCD; produce (S. 73. 3. *cor.*) CB to E, so that $\overline{CE} \times \overline{EB}$ may be equal to $\overline{AB} \times L$; lastly upon $\overline{BE}$ describe (E. 46. 1.) the square EFGB: Then, $\overline{AB}$ is divided in G, so $\overline{BG}^2 = \overline{AG} \times \overline{L}$.

For produce $\overline{FG}$ to H; then (*constr.*) the rectangle $\overline{CE} \times \overline{EB}, = \overline{AB} \times \overline{L}$; but CF is the rectangle $\overline{CE} \times \overline{EB}$, because EF = EB; and CA is the rectangle $\overline{AB} \times \overline{L}$, because CB was made equal to L; ∴ the rectangle CF = CA; take away the common part CG, and there remains BF = HA; and BF is the square of BG, and HA $= \overline{AG} \times L$, because (*constr.* and E. 24. 1.) $\overline{AD} = \overline{BC}$, which was made equal to L.

Prop. LXXXII.

107. Theorem. *If a given circle be cut by any number of circles, which all pass through the same two given points without the given circle, the straight lines, joining the points of each of these intersections, are either all parallel, or all meet when produced in the same point.*

Let CDF be a given circle; and, first, let the



circle ACDB, which passes through the two given points A and B, cut the circle CDF in C and D; let the straight line joining C, D, be parallel to $\overline{AB}$; then shall the straight line joining the points, in which any other circle that passes through A, B, cuts the circle CDF, be parallel to $\overline{AB}$ and $\overline{CD}$.

For, find (E. 1. 3.) the centre K of the circle CDF, and from K draw (E. 12. 1.) $\overline{KEX}$ ⊥ to $\overline{CD}$; ∴ (E. 3. 3.) $\overline{KX}$ bisects $\overline{CD}$ at right ∠∠; ∴ (E. 1. 3. *cor.*) the centre of the circle ACDB is in KX, which (*hyp.* and E. 29. 1.) cuts AB at right ∠∠, and ∴ bisects it; the centres, ∴, of all the circles that pass through A and B are (S. 3. 1. *cor.* 3.) in $\overline{KX}$; ∴ (S. 1. 3.) $\overline{KX}$ cuts all the straight lines, which join the intersections of these circles, with the given circle CDF, at right ∠∠; ∴ (E. 28. 1.) the straight lines joining the several pairs of intersections are parallel to one another and to $\overline{AB}$.

But, secondly, let the circle GLMH, which passes through the two given points G, H, cut the given circle CDF in L and M; and let the straight line joining L and M be not parallel to AB; produce, ∴, $\overline{LM}$ to meet $\overline{GH}$ produced in N; and let any other circle GIFH, passing through G and H, cut the circle CDF in I and F; then are the points I, F and N in the same straight line.

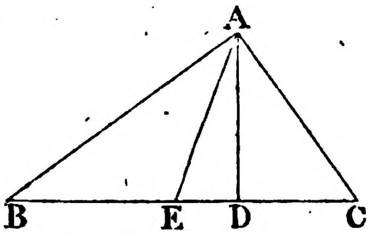
For join N, F, and if $\overline{NF}$, produced, do not pass through I, let it, if it be possible, pass otherwise, as $\overline{NFPQ}$: Then (E. 36. 3. *cor.*) $\overline{PN} \times \overline{NF} = \overline{GN} \times \overline{NH}$; also $\overline{QN} \times \overline{NF} = \overline{LN} \times \overline{NM}$, and $\overline{LN} \times \overline{NM} = \overline{GN} \times \overline{NH}$; ∴ $\overline{QN} \times \overline{NF} = \overline{GN} \times \overline{NH}$; also $\overline{PN} \times \overline{NF} = \overline{GN} \times \overline{NH}$; ∴ $\overline{QN} \times \overline{NF} = \overline{PN} \times \overline{NF}$; ∴ $\overline{QN} = \overline{PN}$; that is the less is equal to the greater, which is impossible; ∴ $\overline{NF}$, when produced, cannot pass otherwise than through the point I, so that the three points I, F and N are in the same straight line.

PROP. LXXXIII.

108. THEOREM. *If a perpendicular be let fall from the right angle, of a right-angled triangle, on the hypotenuse, the rectangle contained by the hypotenuse and either of the segments, into which*

it is divided by the perpendicular, is equal to the square of the side adjacent to that segment.

Let the $\angle$ BAC, of the $\triangle$ ABC, be a right angle,



and from A let $\overline{AD}$ be drawn $\perp$ to the hypotenuse BC: Then $\overline{CB} \times \overline{BD} = \overline{AB}^2$, and $\overline{BC} \times \overline{CD} = \overline{AC}^2$.

For if upon AC, as a diameter, a circle be described, it will pass (S. 29. 1. *cor.* 2.) through the point D, because (*hyp.*) the $\angle$ ADC is a right $\angle$; and (E. 16. 3. *cor.*) it will touch $\overline{AB}$ in A, because the $\angle$ CAB is a right $\angle$; $\therefore$ (E. 36. 3.) $\overline{CB} \times \overline{BD} = \overline{AB}^2$.

And, in the same manner, it may be shewn that $\overline{BC} \times \overline{CD} = \overline{AC}^2$.

Prop. LXXXIV.

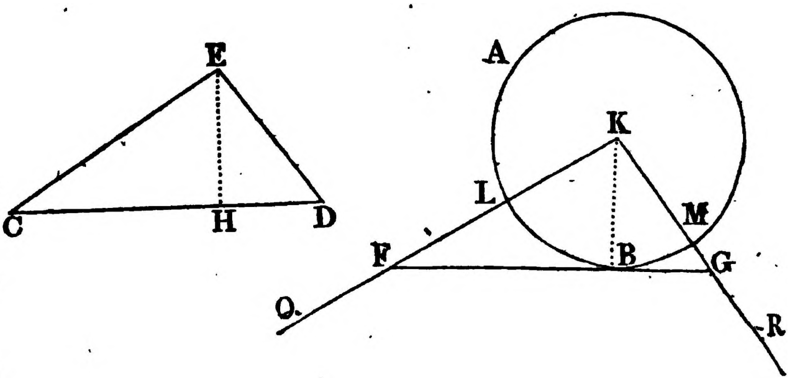
109. Theorem. *To draw a tangent to a circle, such, that the part of it intercepted between two straight lines, given in position, but of indefinite length, shall be equal to a given finite straight line:*

1st, *When the indefinite straight lines both pass through the centre of the circle.*

2dly, *When they are parallel to one another.*

3dly, *When they are not parallel, but are equidistant from the centre.*

Let AB be a given circle, and $\overline{CD}$ a given

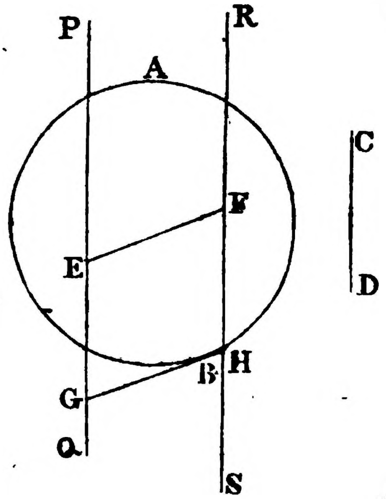


straight line; and first let KQ and KR be two given straight lines, of indefinite length, passing through the centre K of the circle: It is required to draw a straight line, touching the circle AB, so that the part of it intercepted between KQ and KR, shall be equal to CD.

Upon CD describe (S. 61. 3.) a △ CED, having its vertical ∠ CED equal to the given ∠ QKR, and its altitude EH = KB, the semi-diameter of the given circle; from $\overline{KQ}$ cut off $\overline{KF} = \overline{EC}$; and from F draw (E. 17. 3.) the tangent $\overline{FBG}$ to the given circle: Then, the tangent $\overline{FG} = \overline{CD}$.

For let FG touch the circle in B, and join K, B: And since (E. 18. 3.) the ∠ KBF is a right ∠, as is also (*constr.*) the ∠ EHC, and that (*constr.*) EC = KF, and EH = KB, ∴ (S. 73. 1.) the ∠ ECH = ∠ KFB; but (*constr.*) the ∠ CED = ∠ FKG; and the side EC of the △ ECD, is equal to the side KF, of the △ KFG; ∴ (E. 26. 1.) FG = CD.

Secondly, let AB be the given circle, and PQ,

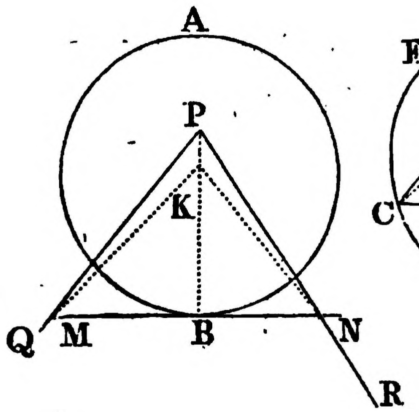


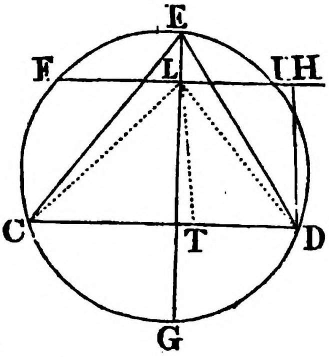
RS, two indefinite but parallel straight lines: It is required to draw a tangent to the circle AB, such that the part of it intercepted between $\overline{PQ}$ and $\overline{RS}$ shall be equal to the given straight line CD.

Take any point E in either of the two parallel straight lines, as PQ, and from the centre E, at a distance equal to $\overline{CD}$, describe a circle cutting $\overline{RS}$

in F; join E, F; ∴ $\overline{EF} = \overline{CD}$; lastly, draw (S. 8. 3:) the straight line GH, touching the circle AB, and parallel to EF; since, ∴, EGHF is a ▭, GH (E. 34. 1.) = EF; and EF was made equal to CD; ∴ the tangent GH = $\overline{CD}$.

Thirdly, let the two indefinite straight lines PQ,





PR, which meet in P, be equi-distant from the centre K, of the circle AB: It is required to draw a straight line touching the circle AB, so that the part of it intercepted between $\overline{PQ}$ and $\overline{PR}$ shall be equal to the given straight line CD.

Join P, K; upon CD describe (E. 33. 3.) a segment of a circle CED, capable of containing an ∠ equal to the given ∠ QPR, and complete the circle CEDG; from D draw (E. 11. 1.) $\overline{DH}$ ⊥ to $\overline{CD}$, and make DH = KB the semi-diameter of the given circle AB; through H draw (E. 31. 1.) $\overline{HIF}$ parallel to $\overline{DC}$; bisect (E. 30. 3.) $\overset{\frown}{FGI}$ in G;

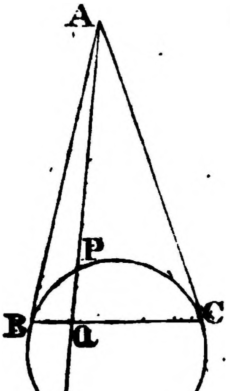
from G draw (S. 75. 3.) $\overline{GLE}$, so that $\overline{LE} = \overline{KP}$; join E, C and E, D; from $\overline{PQ}$ cut off $\overline{PM} = \overline{EC}$; and from M draw (E. 17. 3.) $\overline{MBN}$ touching the circle in B: The tangent MN $= \overline{CD}$.

For join C, L, and D, L, and K, M, and K, N, and K, B; and draw (E. 12. 1.) $\overline{LT} \perp$ to $\overline{CD}$; ∴ LTDH is a ▭, and (E. 34. 1.) $\overline{LT} = \overline{HD}$; and $\overline{HD}$ (*constr.*) $= \overline{KB}$; ∴ $\overline{LT} = \overline{KB}$: Again, because (*constr.* and S. 35. 3.) $\overset{\frown}{CG} = \overset{\frown}{DG}$, ∴ (E. 27. 3.) the ∠ CEG = ∠ DEG; ∴ the ∠ CEL is the half of the ∠ CED; and because (*hyp.*) $\overline{PQ}$ and $\overline{PR}$ are equi-distant from the centre K of the circle AB, ∴ the ∠ QPK, or MPK, is the half of the ∠ QPR, which (*constr.*) is equal to the ∠ CED; ∴ the ∠ MPK = ∠ CEL, and the two sides MP, PK, of the △ PKM, are equal (*constr.*) to the two sides CE, EL of the △ ELC, each to each; ∴ (E. 4. 1.) KM = LC, and the ∠ PMK, = ∠ ECL; and because in the two right-angled △s KBM, LTC, KM = LC, and KB = LT, ∴ (S. 74. 1.) the ∠ KMB = ∠ LCT; and it has been shewn that the ∠ PMK = ∠ ECL; ∴ the whole ∠ PMN is equal to the whole ∠ ECD; also (*constr.*) the ∠ MPN = ∠ CED, and the side PM, of the △ PMN, is equal to the side EC, of the △ ECD; ∴ (E. 26. 1.) $\overline{MN} = \overline{CD}$.

PROP. LXXXV.

110. THEOREM. *If from the intersection of any two tangents to a circle, any straight line be drawn, cutting the chord which joins the two points of contact and again meeting the circumference, it shall be divided by the circumference and the chord into three segments, such, that the rectangle contained by the whole line and the middle part, shall be equal to the rectangle contained by the extreme parts.*

From the intersection A of two straight lines



AB and AC which touch the circle BCR in the points B and C, let there be drawn any straight line APR, cutting the circumference of the circle in P and R, and $\overline{BC}$ in Q: Then $\overline{AR} \times \overline{PQ} = \overline{AP} \times \overline{QR}$.

For since (S. 19. 3. *cor.* 1.) AB = AC, the △ ABC is isosceles, and ∴ (S. 3. 2.) $\overline{AQ}^2 + \overline{BQ} \times \overline{QC} = \overline{AB}^2$; and (E. 35. 3.) $\overline{BQ} \times \overline{QC} = \overline{PQ} \times \overline{QR}$; also (E. 36. 3.) $\overline{AB}^2 = AP \times AR$:

$$\therefore \overline{AQ}^2 + \overline{PQ} \times \overline{QR} = \overline{AP} \times \overline{AR};$$

i. e. (E. 1. 2.) $\overline{AQ} \times \overline{AP} + \overline{AQ} \times \overline{PQ} + \overline{PQ} \times \overline{QR} = \overline{AP} \times \overline{AQ} + \overline{AP} \times \overline{QR}$;

From these equals take away the common rectangle AQ × AP, and there remains $\overline{AQ} \times \overline{PQ} + \overline{QR} \times \overline{PQ} = \overline{AP} \times \overline{QR}$;

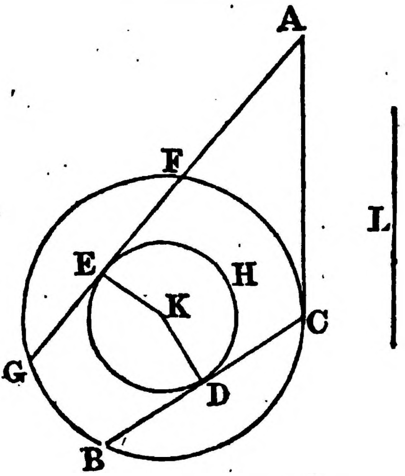
i. e. (E. 1. 2.) $\overline{AR} \times \overline{PQ} = \overline{AP} \times \overline{QR}$.

PROP. LXXXVI.

111. PROBLEM. *To make a rectangle which shall be equal to a given square, and have the difference between its two adjacent sides equal to a given straight line.*

Let AC be the side of a given square, and let L be a given finite straight line: It is required to describe a rectangle which shall be equal to the square of AC, and shall have the difference between its two adjacent sides equal to L.

Describe any circle CBGF capable of containing a straight line equal to L, and having its centre in a ⊥ to $\overline{AC}$ at the point C; the circle CBGF is, ∴ (E. 16. 3.) touched by $\overline{AC}$ in C; from the centre C, at a distance = L, describe a circle cutting



the circumference of CBGF in B, and draw $\overline{CB}$; ∴ $\overline{CB} = \overline{L}$; from the centre of K of the circle CBGF, draw (E. 12. 1.) $\overline{KD} \perp$ to $\overline{BC}$; and from K, as a centre, at the distance KD, describe the circle DEH, which ∴, (E. 16. 3.) touches BC in D; lastly, from A draw (E. 17. 3.) a straight line AFG touching the circle DEH in E, and let AG cut the circumference of CBFG in F and G: Then is the rectangle contained by $\overline{GA}$ and $\overline{AF}$ that which was to be described.

For join K, E; ∴ (*constr.* and E. 18. 3.) the ∠ at E are right ∠; ∴ $\overline{GF}$, which is the difference of $\overline{GA}$ and $\overline{AF}$, is (E. 14. 3.) equal to BC, which was made equal to L; ∴ $\overline{GF} = L$; Also, since $\overline{AC}$ touches the circle CBGF, ∴ (E. 36. 3.) GA $\times$ AF $= \overline{AC}^2$.

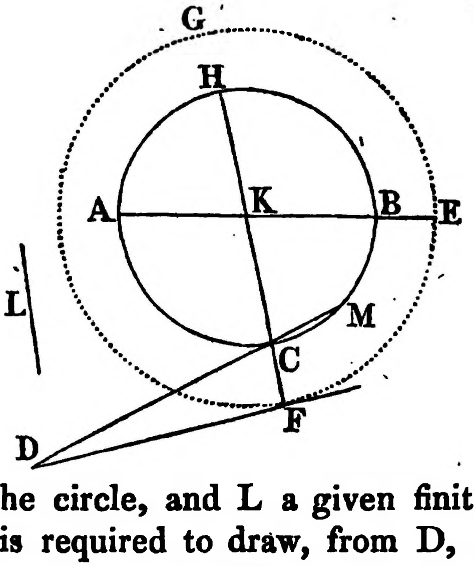
112. Cor. Hence, and from E. 14. 2. a rectangle may be found which shall be equal to a

given rectangle, and which shall have the difference between its two adjacent sides equal to a given straight line.

Prop. LXXXVII.

113. Problem. *From a given point without a circle, to draw a straight line cutting the circle, so that the rectangle contained by the part of it without, and the part within, the circle, shall be equal to a given square.*

Let ABC be a given circle, D a given point without the circle, and L a given finite straight line: It is required to draw, from D, a straight line cutting the circle ABC, so that the rectangle contained by the part of it without, and the part within, the circle, shall be equal to the square of L.



Find (E. 1. 3.) the centre K of the circle ABC; take any diameter AKB, and produce it (S. 67. 3.) to E, so that $\overline{AB} \times \overline{BE}$ = the square of L; from the centre K, at the distance KE, describe the circle EFG; from D draw (E. 17. 3.) the straight line DF touching the circle EFG in F; join K, F, and let $\overline{KF}$ cut the circumference of ABC in C; lastly, draw $\overline{DC}$, and produce it to meet the circumference of ABC again in M: Then shall $\overline{DC} \times \overline{CM}$ be equal to the square of L.

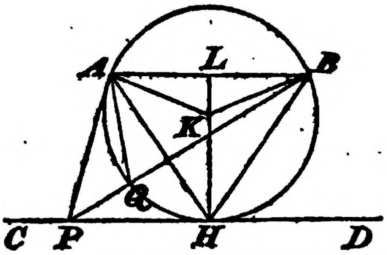
For, produce $\overline{CK}$ to meet the circumference of ABC again in H; then (E. 15. def. 1.) $\overline{HC} = \overline{AB}$, and $\overline{CF} = \overline{BE}$; ∴ $\overline{HC} \times \overline{CF} = \overline{AB} \times \overline{BE}$; but (S. 69. 3.) $\overline{DC} \times \overline{CM} = \overline{HC} \times$ CF, and (*constr.*) $\overline{AB} \times \overline{BE}$ = the square of L; ∴ $\overline{DC} \times \overline{CM}$ = the square of L.

PROP. LXXXVIII.

114. **PROBLEM.** *To describe a circle which shall touch a given straight line, and pass through two given points, both on the same side of the given line, and in the same plane with it.*

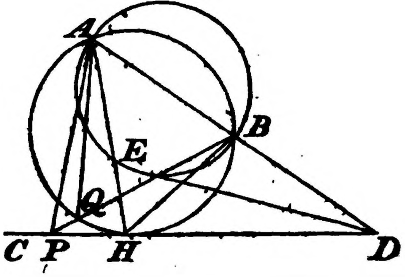
Let CD be a given straight line, and A, B, two given points without it, both on the same side of CD; it is required to draw a circle through A and B, which shall touch CD.

Join A, B; and first, let AB be parallel to CD: Bisect (E. 10. 1.) AB in L; though L draw (E.



11, 1.) LH perpendicular to AB or CD; join A, H; at the point A, in HA, make (E. 23. 1.) the angle HAK equal to the angle AHK, and join K, B; then (E. 6. 1.) KH is equal to KA, and E. 4. 1.) KA is equal to KB; from the centre K, at the distance KA, or KB, or KH, describe the circle AHB; it shall pass through the three points A, H, and B, and (E. 16. 3. *cor.*) shall touch CD in H.

But if AB be not parallel to CD, let AB, produced, meet CD in the point D. Upon AB as a diameter describe the circle ABE, and from D draw (E. 17. 3.) the straight line DE touching it in E; from DC cut off DH (E. 3. 1.) equal to DE, and describe (E. 5. 4.) the circle AHB passing through the three points A, H, and B. The circle AHB, which passes through A and B, touches CD in H.



For (E. 36. 3.) the rectangle contained by AD and DB is equal to the square of DE, and, therefore, is equal also to the square of DH, because DH was made equal to DE; wherefore (E. 37. 3.) the circle AHB touches CD in H.

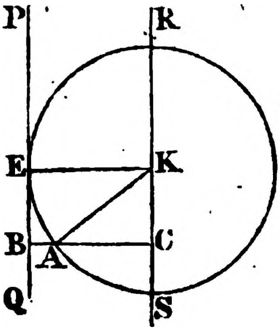
115. Cor. AB subtends a greater angle at the point H, in the straight line CD, than at any other point whatever in CD.

For, let P be any other point in CD; P is without the circle AHB; join A, P, and B, P; let BP cut the circle in Q; also join A, Q. The angle AHB is equal (E. 21. 3.) to the angle AQB; but the exterior angle AQB is greater (E. 16. 1.) than the interior opposite angle APB; wherefore, also, AHB is greater than APB.

Prop. LXXXIX.

116. Problem. *To describe a circle which shall have its centre in a given straight line, which shall pass through a given point, and shall, also, touch another given straight line.*

Let A be a given point, between two given straight lines; and first let the two given straight lines PQ, RS, between which A is posited, be parallel to one another: It is required to describe a circle, which shall have its centre in RS, which shall pass through the given point A, and touch PQ.



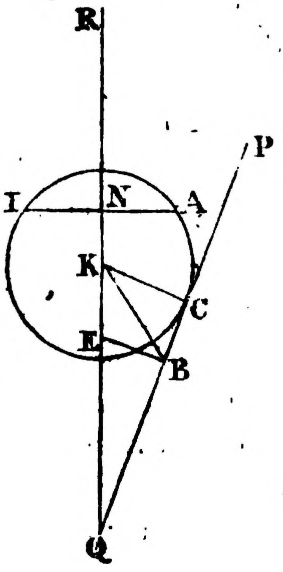
Through A draw (E. 12. 1.) $\overline{BAC}$ ⊥ to $\overline{PQ}$; from A, as a centre, at a distance equal to CB, describe a circle, and let it cut $\overline{RS}$ in K; from K draw KE (E. 31. 1.) parallel to CB; ∴ the figure EC is a ▭, and, ∴ (E. 34. 1. and E. 29. 1.) $\overline{KE}=\overline{CB}$, and the ∠ KEB is a right angle; also, since (*constr.*) $\overline{KA}=\overline{CB}$, and that $\overline{CB}$, as hath been shewn, is equal to $\overline{KE}$, ∴ $\overline{KE}=\overline{KA}$; and ∴ a circle described from K as a centre, at the distance KE, will pass through E and A; and, because the ∠ KEB is a right ∠, it will (E. 16. 3.) touch PQ in E.

Secondly, let K be a given point in the given straight line RQ which meets another given straight line PQ in Q; and let it be required to describe a circle which, having its centre in $\overline{RQ}$, shall pass through K, and which shall touch $\overline{PQ}$.

From K draw (E. 12. 1.) $\overline{KC}$ ⊥ to $\overline{PQ}$; bisect

(E. 9. 1.) the ∠CKQ by $\overline{KB}$, and from B draw (E. 11. 1.) $\overline{BE}$ ⊥ to $\overline{PQ}$; ∴ (E. 28. 1.) $\overline{BE}$ is parallel to $\overline{CK}$; ∴ (E. 29. 1.) the ∠CKB = ∠KFB; but (*constr.*) the ∠EKB = ∠CKB; ∴ the ∠EKB = ∠KBE; ∴ (E. 6. 1.) EK = EB; and ∴ a circle described from the centre E, at the distance EK, will pass through B, and (E. 16. 3.) touch PQ in B, because (*constr.*) the ∠EBQ is a right angle.

Lastly, let the given point A be between two



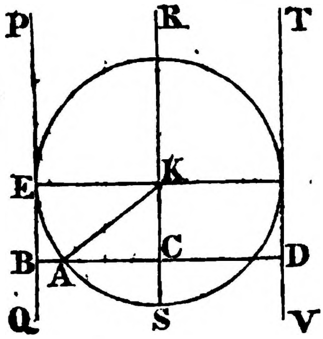
given straight lines PQ and RQ which meet in Q; and let it be required to describe a circle which shall have its centre in RQ, which shall pass through A, and touch $\overline{PQ}$.

From A draw (E. 12. 1.) $\overline{AN}$ ⊥ to $\overline{RQ}$, and produce $\overline{AN}$ to I, so that $\overline{NI} = \overline{NA}$; describe (S. 88. 3.) a circle which shall pass through A and I and touch PQ; and since $\overline{RQ}$ bisects $\overline{AI}$ right $\angle$, the centre of the circle will (E. 1. 3. *cor.*) be in $\overline{RQ}$.

PROP. XC.

117. PROBLEM. *To describe a circle which shall touch two given straight lines, and pass through a given point between them.*

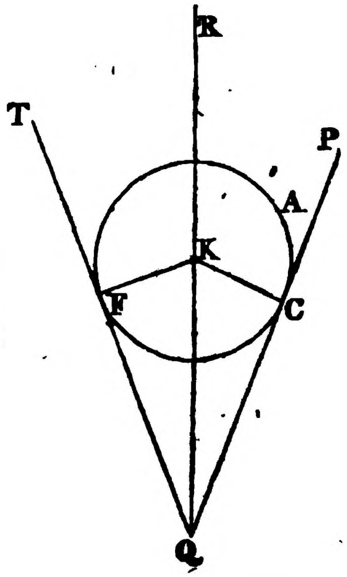
Let A be a given point, between two given straight lines PQ, TV, and, first, let PQ be parallel



to TV: It is required to describe a circle, which shall pass through A and touch both $\overline{PQ}$ and $\overline{TV}$.

Through A draw (E. 12. 1.) $\overline{BAD}$ ⊥ to $\overline{PQ}$, and therefore (E. 29. 1.) also ⊥ to TV; bisect (E. 10. 1.) BD in C, and through C draw (E. 31. 1.) $\overline{RCS}$ parallel to PQ, and ∴ (E. 30. 1.) also parallel to TV; lastly, describe (S. 90. 3.) a circle which shall pass through A and touch PQ: It will also, since its semi-diameter is equal to CB or CD, touch $\overline{TV}$.

Secondly, let the given point A be between two given straight lines TQ, and PQ, which meet in Q: Bisect (E. 9. 1.) the ∠TQP by $\overline{RQ}$, and since (E. 26. 1.) the perpendicular distances of

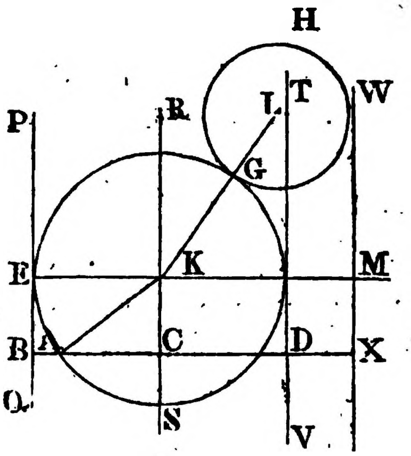


any point in $\overline{RQ}$, from $\overline{TQ}$, and $\overline{PQ}$, are equal to one another, it is manifest, that, if (S. 89. 3.) a circle be described having its centre in RQ, passing through A, and touching either of the two lines TQ, PQ, it will touch the other also.

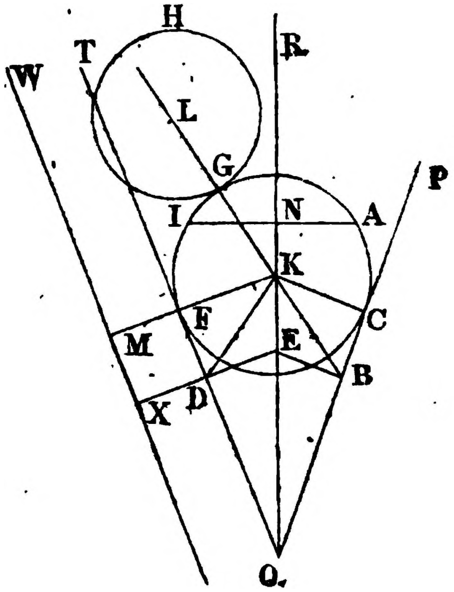
PROP. XCI.

118. **PROBLEM.** *To describe a circle which shall touch two given straight lines, and also touch a given circle, which does not lie wholly without the two given straight lines.*

Let $\overline{PB}$ and $\overline{TD}$ be two given straight lines,



and let HG be a circle, which does not lie wholly without $\overline{PB}$ and $\overline{TD}$: It is required to



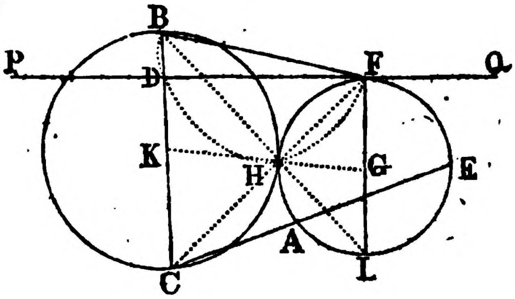
describe a circle which shall touch both $\overline{PB}$ and $\overline{TD}$, and which shall also touch the circle GH.

Find (E. 1. 3.) the centre L of the circle GH; from D draw (E. 11. 1.) $\overline{DX}$ $\perp$ to $\overline{TD}$ and make it equal to the semi-diameter of HG; through X draw (E. 31. 1.) $\overline{XW}$ parallel to $\overline{TD}$; also, as in S. 90. 3. draw $\overline{RK}$, equi-distant from $\overline{PB}$ and $\overline{TD}$; describe (S. 89. 3.) a circle which shall have its centre in $\overline{RK}$, which shall pass through L, and touch $\overline{WX}$; let K be the centre of the circle, so described, and let it touch $\overline{WX}$ in X; join K, M and K, L; and let $\overline{KM}$ and $\overline{KL}$ cut $\overline{TD}$, and the circumference of GH, F and G, respectively: Then, since (E. 18. 3.) the ∠ KMW is a right ∠, and that (*constr.*) $\overline{WX}$ is parallel to $\overline{TD}$, ∴ the ∠ KFT is also a right ∠; and, because (*constr.*) MD is a ▭, (E. 34. 1.) $\overline{MF}=\overline{XD}$; but (*constr.*) $\overline{XD}=\overline{LG}$; ∴ $\overline{MF}=\overline{LG}$; and (*constr.* and E. 15. def. 1.) $\overline{KM}=\overline{KL}$; ∴ $\overline{KF}=\overline{KG}$, and a circle described from the centre K, at the distance KF, will (E. 16. 3. *cor.*) touch TD in F, will pass through G, and (S. 6. 3.) will touch the circle HG in G.

PROP. XCII.

119. PROBLEM. *To describe a circle which shall touch both a given circle, and a given straight line, and which shall, also, pass, first, through a given point without the given circle; and, secondly, through a given point within the circle.*

Let BCH be the given circle, PQ the given straight line, and first let the given point A be without the circle: It is required to describe a



circle which shall pass through A, and which shall touch both $\overline{PQ}$ and the circle BCH.

Find (E. 1. 3.) the centre K of the circle BCH, and draw (E. 12. 1.) the diameter BDKC $\perp$ to PQ; join C, A,* and produce CA to E (S. 67. 3. *cor.*) or divide it (S. 71. 3. *cor.* 3.) so that $\overline{EC} \times \overline{CA} = \overline{BC} \times \overline{CD}$; describe (S. 88. 3.) a circle AEF, which shall pass through A and E, and touch $\overline{PQ}$: It shall also touch the circle BCH.

For, let the circle AEF touch $\overline{PQ}$ in F; find its centre G; and draw the diameter FGL, which (*constr.* E. 18. 3. and E. 28. 1.) is parallel to $\overline{BC}$; join, B, F, and C, F; and let $\overline{CF}$ cut the circumference of BCH in H; join, also, B, H and F, H and K, H; and let $\overline{KH}$ meet $\overline{FL}$ in G; upon BF as a diameter, describe the circle BDHF, which, because the ∠∠ BDF, BHF (*constr.* and E. 31. 3.) are right ∠∠, will pass (S. 29. 1. *cor.* 2.) through D and H; ∴ (E. 36. 3. *cor.*) $\overline{BC} \times \overline{CD} = \overline{FC} \times \overline{CH}$; but (*constr.*) $\overline{BC} \times \overline{CD} = \overline{EC} \times \overline{CA}$; ∴ $\overline{FC} \times \overline{CH} = \overline{EC} \times \overline{CA}$, and, ∴, the point H is in the circumference of the circle AEF; otherwise (E.

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* If $\overline{AC}^2 > \overline{BC} \times \overline{CD}$, then $\overline{CA}$ must be divided into two parts, so that the rectangle contained by AC and the segment toward C shall be equal to $\overline{BC} \times \overline{CD}$. Also, in this application of S. 67. 3, CA must first be produced, so that the rectangle contained by CA and the *part* produced, shall be of the given magnitude; and then from the whole line, CE must be cut off equal to the part produced.

36. 3. *cor.*) the greater of two rectangles would be equal to the less. The point is, ∴, common to both the circles AEF, BCH.

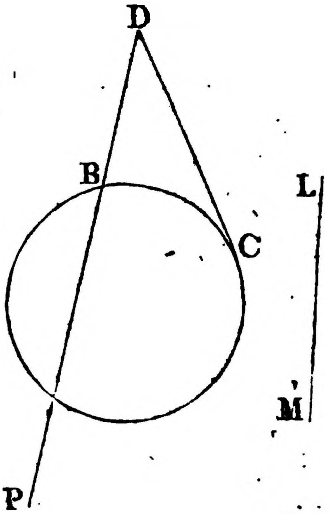
And, since (*constr.*) $\overline{FL}$ is parallel to $\overline{BC}$, ∴ (E. 29. 1.) the ∠ GFH = ∠ HCB; but, since (*constr.* and E. 31. 3.) the ∠ BHC is a right ∠, the ∠ HCB + ∠ CBH = (E. 32. 1.) a right ∠; ∴ the ∠ GFH + ∠ KBH = a right ∠; that is (E. 15. def. 1. and E. 5. 1.) the ∠ GFH + ∠ KHB = a right ∠; and (*constr.* and E. 31. 3.) the ∠ BHF is a right ∠; ∴ (E. 13. 1.) the ∠ KHB + ∠ GHF = a right ∠; ∴, the ∠ GFH = ∠ GHF, and (E. 6. 1.) GF = GH: But G is in the diameter of the circle AEF; ∴ (E. 7. 3.) G is the centre of the circle AEF, which ∴ (S. 6. 3.) touches the circle BCH in H.

And, in a similar manner, the problem may be solved, when it admits of a solution, if the given point be within the given circle: It is manifest, however, that, in this latter case, the given straight line which is to be touched cannot lie wholly without the given circle.

PROP. XCIII.

120. PROBLEM. *In a straight line of indefinite length, but given in position, which cuts a given circle, to find a point, from which if a straight line be drawn to touch the circle, it shall be equal to a given finite straight line.*

Let LM be a given finite straight line, PAB a straight line given in position, but indefinite in length, cutting the given circle ABC in A and B: It is required to find a point in PB, from which, if a tangent be drawn to the circle ABC, it shall be equal to L.

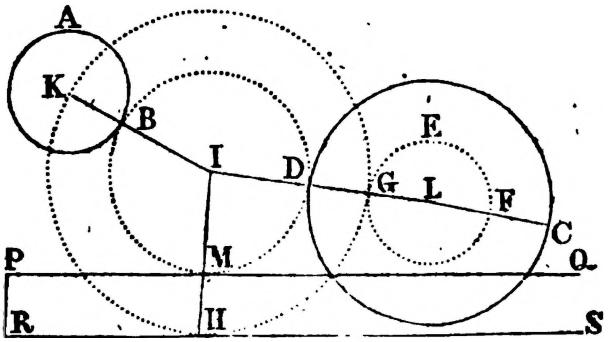


Produce (S. 73. 3.) AB to D so that $\overline{AD}\times\overline{DB}=\overline{LM}^2$; and from the centre D, at a distance = LM, describe a circle cutting the circumference of the circle of ABC in C; draw $\overline{DC}$; ∴ $\overline{DC}=\overline{LM}$; ∴ but (*constr.*) $\overline{AD}\times\overline{DB}=\overline{LM}^2$; ∴ $\overline{AD}\times\overline{DB}=\overline{DC}^2$; ∴ (E 37. 3.) DC touches the circle ABC, in C; and (*constr.*) it is equal to LM, and is drawn from a point D in the given indefinite straight line PAB.

PROP. XCIV.

121. **PROBLEM.** *To describe a circle that shall touch a given straight line, and that shall also touch two given circles.*

Let AB and CD be the two given circles, and



PQ a given straight line; and first, let neither of the two given circles lie within the other: It is required to describe a circle which shall touch both the given circles AB and CD, and which shall also touch $\overline{PQ}$.

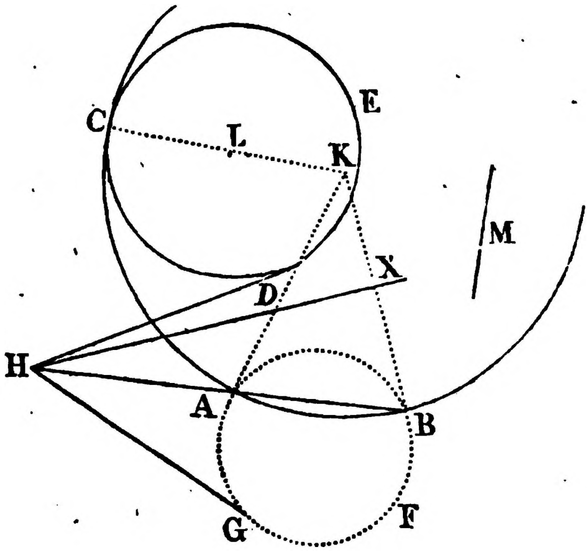
Find (E. 1. 3.) the centres K and L of the circles AB and CD; and if the circles be unequal, let CD be the greater; from any semi-diameter, as LC, of the greater, cut off CF equal to a semi-diameter of the less circle; from the centre L, at the distance LF describe the circle FGE; from any point P, in $\overline{PQ}$, draw (E. 11. 1.) $\overline{PR}$ ⊥ to $\overline{PQ}$, and make $\overline{PR}$ also equal to the semi-diameter of the less circle AB; through R draw (E. 31. 1,) RS parallel to PQ; describe (S. 92. 3.) a circle KHG, passing through the point K, touching $\overline{RS}$, in H, and touching the circle EGF, in G; let I be the centre of the circle KHG; join K, I, and L, I, and I, H; ∴ (E. 18. 3.) the ∠IHR is a right ∠, and ∴ (*constr.* and E. 29. 1.) the exterior ∠ IMP is, also, a right ∠; and the figure PH is a ▭; ∴ (E. 34. 1.) $\overline{MH} = \overline{PR}$; and (*constr.*) $\overline{PR} = \overline{KB}$ or $\overline{DG}$; ∴ $\overline{MH} = \overline{KB}$ or $\overline{DG}$; ∴ $\overline{IB}$, $\overline{IM}$ and $\overline{ID}$ are all equal; and (E. 16. 3. *cor.* and S. 6. 3). a circle described from the centre I, at the distance IM, will touch $\overline{PQ}$ in M, the circle AB in B, and the circle CD in C.

But if the two circles AB, CD, be equal to one another, find, as before, their centres K, and L, and draw $\overline{RS}$ at a perpendicular distance from $\overline{PQ}$ equal to the semi-diameter of AB or CD: Then, if (S. 88. 3.) a circle be described passing through K and L, and touching $\overline{RS}$, it is evident, that its centre will be the centre of the circle which is to be described, and its semi-diameter will be found, as in the former case, by joining that centre and the centre of either of the two equal and given circles.

And, in a similar manner, the problem may be solved, when it admits of a solution, if the two given circles do not lie without one another.

PROP. XCV.

122. **PROBLEM.** *To describe a circle which shall touch a given circle, and pass through two given points, either both without the circle, or both within it.*



Let A, B, be two given points, and CDE a given circle: It is required to describe a circle which shall pass through A and B, and which shall also touch the circle CDE.

First, let the two given points, A and B, be without the circle CDE: And if A and B be equally distant from the centre of CDE, it is manifest (S. 6. 3.) that a circle described (S. 5. 1. *cor.*) so as to pass through the two given points, and through the extremity of a diameter of the given circle drawn perpendicular to the straight line joining those points, will touch the given circle.

But if the points, A and B, be not equally distant from the centre of the circle CDE, take any point F, without the circumference of CDE, and through A, B and F, describe (S. 5. 2. *cor.*) the circle AFB; draw (S. 79. 3.) $\overline{HX}$, so that the straight lines which are drawn from any point of it, touching the two circles CDE, AGB, shall be equal to one another, and let $\overline{BA}$, produced, meet $\overline{HX}$ in H; from H draw (E. 17. 3.) $\overline{HC}$ and $\overline{HD}$, touching the circle CDE in C and D; lastly, describe (S. 5. 1. *cor.*) two circles, the one passing through B, A, and C, and the other through B, A, D; the circles so described shall touch the given circle CDE, in the points C and D, respectively.

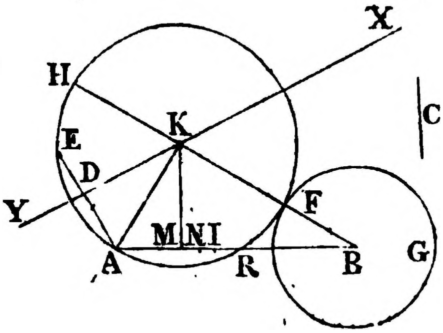
For, from H draw (E. 17. 3.) $\overline{HG}$, touching the circle AGB in G: Then (E. 36. 3.) $\overline{BH} \times \overline{HA} = \overline{HG}^2$; but (*constr.*) $\overline{HG} = \overline{HC}$; $\therefore$ $\overline{BH} \times \overline{HA} = \overline{HC}^2$; $\therefore$ (E. 37. 3.) $\overline{HC}$ touches the circle described through B, A and C; and (*constr.*) it also touches the circle CDE; $\therefore$ (E. 3. def. 3.) the circle BAC which passes through A and B, touches the circle CDE in C.

In the same manner it may be shewn, that the circle described so as to pass through A, B and D, touches the circle CDE in D: And by a like construction may the problem be solved, when the two given points are both within the given circle.

Prop. XCVI.

123. **Problem.** *To find a point in a straight line, given in position, from which if two straight lines be drawn to two given points, without the given line, they shall have, first, their difference, and, secondly, their aggregate, equal to a given finite straight line.*

Let A, B be two given points, XY a straight



line of indefinite length, but given in position; and let C be a given finite straight line: It is required to find a point in $\overline{XY}$, from which if two straight lines be drawn to A and B, they shall have, first, their difference equal to C.

From A draw (E. 11. 1.) $\overline{AD} \perp$ to $\overline{XY}$; produce $\overline{AD}$ to E, and make $\overline{DE} = \overline{AD}$; from the centre B, at a distance equal to C, describe the circle FG; also, describe (S. 95. 3.) a circle EAF, which shall pass through E and A, and which shall touch the circle GF, in F; let K be the centre of the circle EAF, which centre (E. 1. 3. *cor.*) is in $\overline{XY}$: Then is K the point which was to be found.

For join K, A and K, B; ∴ (E. 11. 3. or E. 12 3.) $\overline{KB}$ passes through the point of contact F; and (E. 15. def. 1.) $\overline{KA}=\overline{KF}$; ∴ $\overline{KB}-\overline{KA}=\overline{BF}$; and (*constr.*) $\overline{BF}=\overline{C}$; ∴ $\overline{KB}-\overline{KA}=\overline{C}$.

And, by a like construction, may a point be found in $\overline{XY}$, from which if two straight lines be drawn, to A and B, their aggregate shall be equal to a given straight line.

But, in this case, the two points A and E must fall within the circle described from the centre B, at a distance equal to that given line; otherwise, the problem is impossible.

124. Cor. 1. Let AB be (E. 10. 1.) bisected in I, let (E. 12. 1.) KM be drawn ⊥ to AB, and let the circumference EAF cut AB in A and R, and BK produced in H: Then it is manifest, (*constr.* and E. 3. 3.) that 2IM = BR; and (E. 36. 3. *cor.*) $\overline{AB}\times\overline{BR}=\overline{HB}\times\overline{BF}$; *i. e.* $2\overline{AB}\times\overline{IM}=\overline{HB}\times\overline{BF}$, or $\overline{HF}\times\overline{BF}+\overline{BF}^2$ (E. 3. 2.)

Let now IN be taken in IM (S. 67. 3.) so that $2\overline{AB}\times\overline{IN}=\overline{BF}^2$; ∴, if $2\overline{AB}\times\overline{IN}$ be taken from $2\overline{AB}\times\overline{IM}$, and if $\overline{BF}^2$ be taken from $\overline{HF}\times\overline{BF}+\overline{BF}^2$, there will remain $2\overline{AB}\times\overline{NM}=\overline{HF}\times$

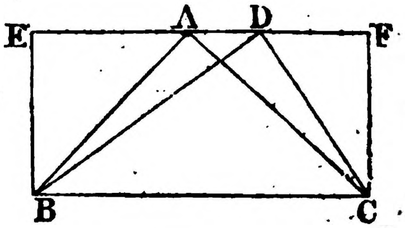
$\overline{BF}$ or $2\overline{AK}\times\overline{BF}$; $\therefore \overline{AB}\times\overline{NM}=\overline{AK}\times\overline{BF}$.

125. Cor. 2. There is only one point K, in YX, from which if straight lines be drawn to A and B, their difference shall be equal to the given line C.

Prop. XCVII.

126. Problem. *The base and the altitude of a triangle being given, together with the aggregate or the difference, of the two remaining sides, to construct the triangle.*

Let $\overline{BC}$ be the given base of a $\triangle$, and BE, drawn



$\perp$ to $\overline{BC}$, equal to its given altitude: It is required to construct a $\triangle$, which shall have $\overline{BC}$ for its base, its altitude equal to $\overline{BE}$, and, first, the aggregate of its two remaining sides of a given length.

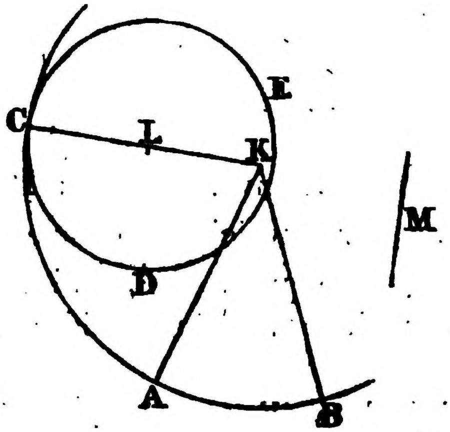
Through E draw (E. 31. 1.) $\overline{EF}$ parallel to $\overline{BC}$; find (S. 96. 3.) a point A from which, if $\overline{AB}$ and $\overline{AC}$ be drawn, their aggregate shall be equal to the given aggregate; if, $\therefore$ A, B and A, C be joined, it is manifest that ABC is the △, which was to be described.

And, in the same manner, by the help of S. 96. s., may the problem be solved if the difference, instead of the aggregate of the two sides of the △, be given.

PROP. XCVIII.

187. PROBLEM. *Three points being given, to find a fourth, from which if straight lines be drawn to the other three, two of them shall be equal, and the difference between either of these and the third shall be equal to a given straight line.*

Let A, B and L be three given points, and M



a given finite straight line: It is required to find a fourth point, from which, if three straight lines be drawn to L, A, and B, two of them shall be equal, and the difference between either of these and the third shall be equal to $\overline{M}$.

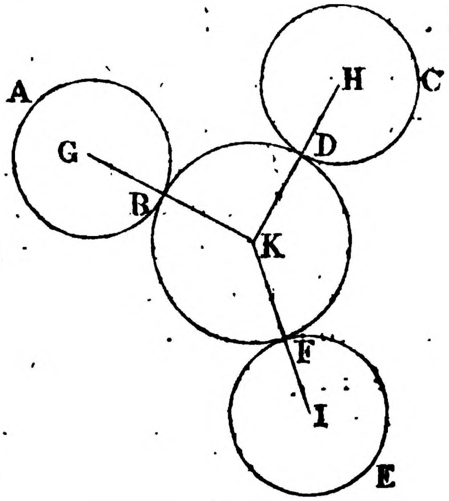
From L as a centre, at a distance equal to M, describe the circle CDE; describe (S. 95. 3.) a circle CAB, which shall pass through A and B, and which shall touch the circle CDE; and let K be the centre of the circle CAB: Then is K the point which was to be found.

For join K, L; ∴, (E. 11. 3. or E. 12. 3.) $\overline{KL}$, produced, passes through the point C, in which the two circles CDE, CAB touch one another; join, also, K, A and K, B; ∴ (E. 15. def. 1.) $\overline{KA}$, KB and KC are equal to one another; and $\overline{KL} = \overline{KC} - \overline{LC}$; but (*constr.*) $\overline{LC} = M$; ∴ KL is equal to the difference between KC and M, that is, to the difference between $\overline{KA}$, or $\overline{KB}$ and M.

Prop. XCIX.

128. Problem. *To describe a circle that shall touch three given circles, of which two are equal to one another.*

Let AB, CD, EF be three given circles, of which the two AB and CD are equal to one another: It is required to describe a circle which shall touch the three given circles AB, CD, and EF.



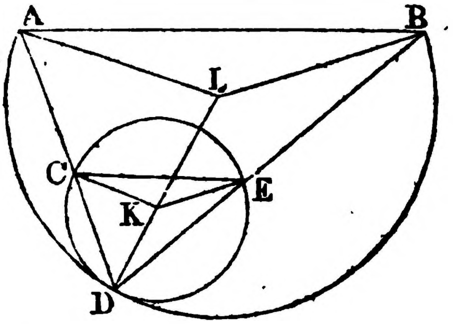
Find (E. 1. 3.) the centres G, H and I of the three given circles; find, also, (S. 98. 3.) a point K, the distances of which from G and H, shall be equal to one another, and shall either of them differ from the distance between the points K and I, by the semi-diameter of the given circle EF; if, then, $\overline{GK}$, $\overline{HK}$ and $\overline{IK}$ be drawn, it is manifest that KB, KD, and KF, are all equal to one another, and, ∴, that a circle, BDF, described from the centre K at the distance KB, will pass through B, D and F, and (S. 6. 3.) will touch the circles AB, CD, and EF in the points B, D and F, respectively.

PROP. C.

129. **PROBLEM.** *To find a point, in the circumference of a given circle, from which if two*

straight lines be drawn to two given points, without the circle, the chord joining the intersections of the lines so drawn and the circumference, shall be parallel to the straight line joining the two given points.

Let CDE be a given circle, and A, B, two



given points without it: It is required to find a point in the circumference of CDE, from which if two straight lines be drawn to A and B, the chord joining their intersections with the circumference of CDE shall be parallel to $\overline{AB}$.

Find (E. 1. 3.) the centre K of the circle CDE; find, also, (S. 98. 3.) a point L, the distances of which from A and B, shall be equal to one another, and shall, either of them, differ from the distance between L and K, by the semi-diameter of the given circle CDE; join L, A and L, B and L, K, and produce LK to meet the circumference of CDE, in D: Then is D the point which was to be found.

For (*constr.*) $\overline{LD}$ is equal to $\overline{LA}$ or $\overline{LB}$; and a circle, ADB described from L, as a centre, at the distance LA, will (S. 6. 3.) touch CDE in D, and will pass through B; draw $\overline{DA}$ and $\overline{DB}$, cutting the circumference of CDE in C and E; join, likewise, C, E, and K, C and K, E: And since (*constr.*) the ∠CKE is an ∠ at the centre, and the ∠CDE is an ∠ at the circumference of the circle CDE, the ∠CKE is (E. 20. 3.) the double of the ∠CDE; in the same manner, it may be shewn that the ∠ALB is the double of the ∠ADB or CDE; ∴ the ∠CKE = ∠ALB; ∴ (E. 32. 1. and E. 5. 1.) the ∠∠ KEC, LBA, at the bases of the isosceles △s CKE, ALB, are equal to one another: Again, since (E. 15. def. 1.) the △s EKD, BLD are isosceles, the ∠KDE = ∠KED, and the ∠LDB or KDE = ∠LBD; ∴ the ∠KED = ∠LBD; and it has been shewn that the ∠KEC = ∠LBA; ∴ the whole ∠CED = the whole ∠ABD; ∴ (E. 28. 1.) $\overline{CE}$ is parallel to $\overline{AB}$.

A SUPPLEMENT TO THE ELEMENTS OF EUCLID.

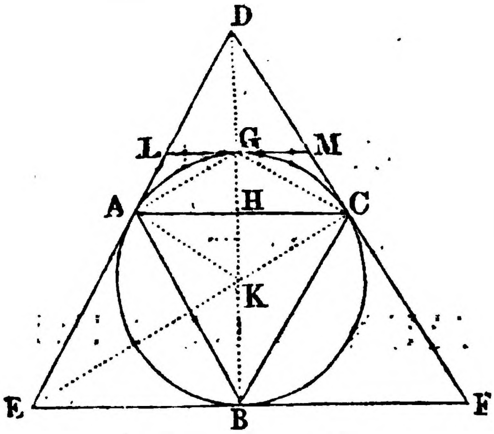
BOOK IV.

PROP. I.

1. THEOREM. *If an equilateral triangle be described about a given circle, the straight lines joining the points of contact shall contain another equilateral triangle; and the side of the circumscribed triangle is the double of the side of the inscribed triangle so contained.*

Let ABC be a given circle: About it describe (E. 1. 1. and E. 3. 4.) the equilateral △ DEF, the sides of which touch the circle in the points A, B and C, respectively; draw $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$: Then is $\overline{ABC}$ an equilateral △, and any side, as EF, of the △ DEF, is the double of any side, as AC, of the △ ABC.

For (*constr.* E 5. 1. *cor.* E. 32. 1.) each of the



$\angle\angle$ D, E, F, is the third of two right $\angle\angle$; $\therefore$ (S. 19. 1. E. 5. 1. E. 32. 1.) each of the $\angle\angle$ of the $\triangle\triangle$ DAC, EAB, FBC, is the third of two right $\angle\angle$, and they are all equal to one another; $\therefore$ (E. 32. 3.) the $\triangle$ ABC is equiangular; and, $\therefore$, (E. 6. 1. *cor.*) it is also equilateral.

Again, since it has been shewn that $\overline{AB}=\overline{AC}$, $\therefore$ (E. 28. 3.) $\overset{\frown}{AB}=\overset{\frown}{AC}$; $\therefore$ (S. 60. 3.) $\overline{DE}$ is parallel to $\overline{CB}$; and in the same manner it may be shewn, that $\overline{EF}$ is parallel to $\overline{AC}$, and $\overline{DF}$ parallel to $\overline{AB}$; $\therefore$ the figures ACBE; ACFB are ▱; $\therefore$ (E. 34. 1.) $\overline{AC}=\overline{EB}$; also $\overline{AC}=\overline{BF}$; $\therefore$ $\overline{EB}+\overline{BF}$, that is $\overline{EF}$, is the double of $\overline{AC}$.

2. Cor. 1. If K be the centre of the circle, and if K, and any angular point of the circumscribed equilateral $\triangle$, as D, be joined, $\overline{DK}$ is

bisected in G, by the arch AGC, and $\overline{KG}$ is bisected in H, by the side AC of the inscribed equilateral △.

For draw $\overline{AG}$, $\overline{AK}$, $\overline{CG}$, $\overline{CK}$, and produce $\overline{DK}$ to meet $\overline{EF}$: Then since (S. 19. 3. *cor.* 1.) $\overline{DA} = \overline{DC}$, and (E. 15. def. 1.) $\overline{AK} = \overline{CK}$, ∴ (S. 1. 3. *cor.*) $\overline{DK}$ and $\overline{AC}$ bisect one another at right ∠∠ in H; and the ∠ ADK = ∠ CDK; also $\overline{ED} = \overline{FD}$; ∴ (E. 4. 1.) $\overline{DK}$ produced bisects $\overline{EF}$, and ∴ passes through the point of contact B.

Again (E. 32. 3.) the ∠ EAB = ∠ AGB or AGK; and (E. 5. 1.) the ∠ KAG = ∠ AGK; ∴ (E. 32. 1.) the ∠∠ of the △ AKG are equal to the ∠∠ of the △ ABE, which in the proposition was shewn to be equilateral and ∴ equiangular; ∴ (E. 6. 1. *cor.*) the △ AKG is equilateral; ∴ $\overline{AG} = \overline{AK}$; and in the same manner, it may be shewn that $\overline{CG} = \overline{CK}$; ∴ (S. 1. 3. *cor.*) $\overline{HG} = \overline{HK}$; and it has been shewn that $\overline{HD} = \overline{HB}$; from these equals take the equals $\overline{HG}$ and $\overline{HK}$, and there remains $\overline{GD}$ equal to $\overline{KB}$ or $\overline{KG}$; ∴ $\overset{\frown}{AGC}$ bisects $\overline{DK}$ in G, and $\overline{AC}$ bisects $\overline{KG}$ in H.*

3. Cor. 2. A straight line which touches a circle, at the extremity of a diameter drawn from

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* From this corollary may be derived an easy practical method of inscribing an equilateral triangle in a given circle, and of describing an equilateral triangle about a given circle.

the point of contact of any side of an equilateral △ described about the circle, and which is terminated by the two remaining sides, is the side of an equilateral and equiangular hexagon described about the circle.

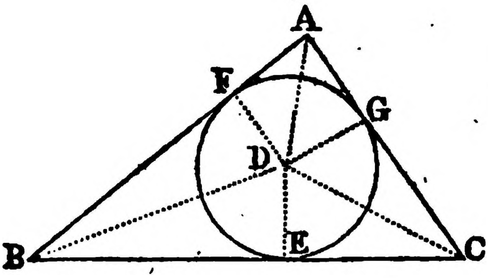
For, from the point B, in which the side EF, of the equilateral △ DEF, touches the circle ABC, let the diameter BG be drawn, which, as hath been shewn, passes through D, and draw (E. 17. 1.) $\overline{LM}$ touching the circle in G; draw, also, $\overline{AG}$ and $\overline{CG}$: Then, since it has been proved (*cor.* 1.) that AG and CG are each of them equal to the semi-diameter of the circle, ∴ (E. 15. 4.) they are the sides of an equilateral and equiangular hexagon inscribed in the circle: And if two other tangents be drawn at the extremities of the diameters which pass through the two points A and C, the remaining points of contact may, in the same manner, be shewn to be the remaining angular points of the inscribed hexagon of which AG and GC are sides: And in the same manner as the pentagon described about a circle is proved, in E. 12. 4. to be equilateral and equiangular, may the hexagon thus described about the circle ABC be shewn to be equilateral and equiangular.

4. Cor. 3. An equilateral triangle inscribed in a given circle is a fourth part of the equilateral triangle described about that circle.

PROP. II.

5. THEOREM. *If a triangle be described about a given circle, the rectangle contained by the perimeter of the triangle and the semi-diameter of the circle shall be double of the triangle.*

Let FEG be a given circle: Describe (E. 8. 4.)



any △, ABC, the sides of which touch the circle in the points F, E, G: The rectangle contained by the semi-diameter of the circle, and the perimeter of the △ ABC, is double of the △ ABC.

For take D the centre of the circle FEG, and draw $\overline{DF}$, $\overline{DG}$, $\overline{DE}$, $\overline{DA}$, $\overline{DB}$ and $\overline{DC}$: Then (E. 41. 1.) the rectangle contained by $\overline{DF}$ and $\overline{AB}$ is double of the △ ADB, the rectangle contained by $\overline{DE}$ and $\overline{BC}$ is double of the △ BDC, and the rectangle contained by $\overline{DG}$ and $\overline{AC}$ is double of the △ ADC; but (E. 15. def. 1.) $\overline{DF}$, $\overline{DE}$ and $\overline{DG}$ are equal to one another; if, ∴, $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, be supposed to be placed in the same straight line, the rectangle contained by their aggregate and a semi-diameter of the circle FEG, is (E. 1. 2.) double of the three △ ADB, BDC, CDA, that is, of the whole △ ABC.

6. Cor. 1. If any number of △ be described about a given circle they shall be equal to one another.

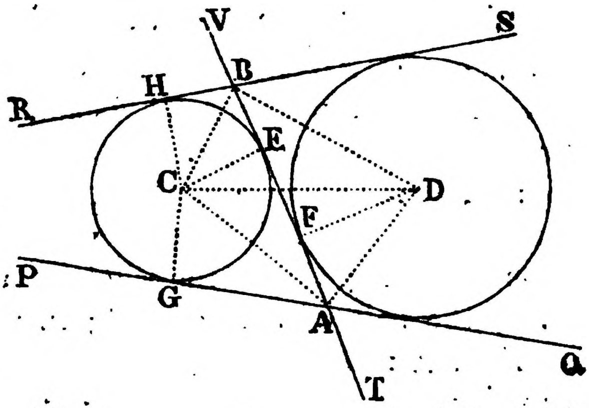
7. Cor. 2. In the same manner it may be shewn that the rectangle contained by the perimeter of any rectilineal figure described about a given circle and the semi-diameter of the circle is double of the rectilineal figure: And, therefore, all rectilineal figures deseribed about the same circle that have equal perimeters, are equal to one another.

Prop. III.

8. Problem. *Three straight lines being given, which, when produced, do not all three meet in the same point, and of which the middle line is not parallel to either of the others, to describe a circle which shall touch each of them.*

Let PQ, RS, TV, be three given straight lines, which, when produced, do not all meet in the same point: It is required to describe a circle which shall touch $\overline{PQ}$, $\overline{RS}$ and $\overline{TV}$.

Let $\overline{PQ}$ and $\overline{RS}$ cut $\overline{TV}$ in A, and B; bisect (E. 9. 1.) the ∠ PAB, ABR, QAB, ABS, by



$\overline{AC}$, $\overline{BC}$, $\overline{AD}$ and $\overline{BD}$, and let $\overline{AC}$ and $\overline{BC}$ meet in C, and $\overline{AD}$ and $\overline{BD}$ in D; from C and D draw (E. 12. 1.) $\overline{CE}$ and $\overline{DF}$ $\perp$ to $\overline{AB}$: Then shall a circle described from the centre C at the distance CE, touch $\overline{AB}$ and $\overline{AP}$ and $\overline{BR}$; and a circle described from the centre D, at the distance DF, shall touch $\overline{AB}$ and $\overline{AQ}$ and $\overline{BS}$.

For draw (E. 12. 1.) from C, $\overline{CG}$ $\perp$ to $\overline{AP}$, and $\overline{CH}$ $\perp$ to $\overline{BR}$, and join C, A and C, B: And because (*constr.*) the $\angle$EAC = $\angle$GAC, and the ∠∠ at E and G are right ∠∠, and that $\overline{AC}$ is common to the two triangles AEC, AGC, ∴ (E. 26. 1.) $\overline{CG} = \overline{CE}$; and in the same manner it may be shewn that $\overline{CE} = \overline{CH}$; ∴ $\overline{CE}$, $\overline{CG}$, and $\overline{CH}$ are equal to one another; and ∴ a circle described from the centre C at the distance CE will pass through G and H, and (*constr.* and E. 16. 3. *cor.*) will touch $\overline{AB}$ in E, $\overline{AP}$ in G, and $\overline{BR}$ in H.

In the same manner it may be proved that a circle described from the centre D, at the distance $\overline{DF}$, will touch $\overline{AB}$, $\overline{AQ}$ and $\overline{BS}$.

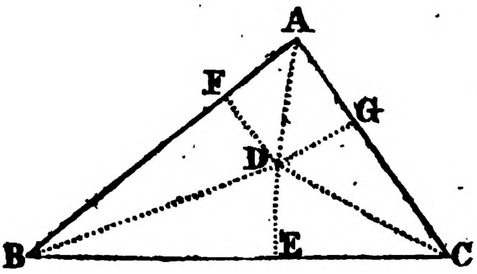
9. **Cor.** The four points A, C, B and D are in the circumference of a circle.

For join C, D: The two ∠s CAE, DAE, together, are (*constr.*) the half of the two ∠s PAB, QAB taken together; that is the whole ∠ CAD is (E. 13. 1.) the half of two right ∠s; ∴ the ∠ CAD is a right ∠: In the same manner it may be shewn that the ∠ CBD is a right ∠; ∴ (S. 29. 1. *cor.* 2.) a circle described upon $\overline{CD}$ as a diameter, will pass through A and B.

Prop. IV.

10. **Theorem.** *The three straight lines, which bisect the three angles of a triangle, meet in the same point.*

Let ABC be a given triangle: The three



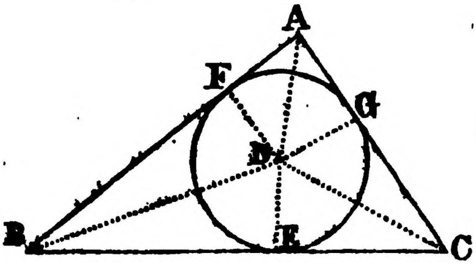
straight lines which bisect its ∠s, meet in the same point.

For (E. 9. 1.) bisect the ∠∠ ABC, ACB, by $\overline{BD}$ and $\overline{CD}$, which meet in D, and join A, D; also from D draw (E. 12. 1.) $\overline{DE}$ ⊥ to $\overline{BC}$, $\overline{DF}$ ⊥ to $\overline{AB}$, and $\overline{DG}$ ⊥ to $\overline{AC}$: Then it may be shewn as in the next preceding proposition, that $\overline{DF} = \overline{DG}$; and $\overline{DA}$ is common to the two right-angled △△ AFD, AGD; ∴ (S. 74. 1.) the ∠ FAD = ∠ GAD; ∴ $\overline{AD}$ bisects the ∠ BAC, and the three straight lines which bisect the three ∠∠ of the △ ABC meet in the same point D.

PROP. V.

11. THEOREM. *If a circle be inscribed in a right-angled triangle, the excess of the two sides, containing the right angle, above the third side, is equal to the diameter of the inscribed circle.*

Let ABC be a △ having one of its ∠∠ BAC, a

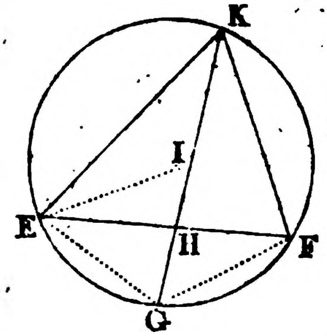


right ∠; and let (E. 4. 2.) the circle FEG, of which D is the centre, be inscribed in it. The excess of AB + AC above BC is equal to the diameter of the circle FEG.

For join the centre D, and the points of contact E, F, and G; join, also, D, A: Then since (S. 19. 3. *cor.* 1.) $\overline{BE} = \overline{BF}$, and $\overline{CE} = \overline{CG}$, it is evident that $\overline{AF} + \overline{AG}$, or $2\overline{AF}$, is the excess of $\overline{AB} + \overline{AC}$ above $\overline{BC}$: Again, since $\overline{AF} = \overline{AG}$, and $\overline{FD} = \overline{GD}$, and $\overline{AD}$ is common to the two △ AFD, AGD, ∴ (E. 8. 1.) the ∠ FAD = ∠ GAD; but (*hyp.*) the ∠ FAG is a right ∠; ∴ the ∠ FAD is half of a right ∠; also (*constr.* and E. 18. 3.) the ∠ AFD is a right ∠; ∴ (E. 32. 1.) the ∠ FDA is half of a right ∠; ∴ the ∠ FAD = ∠ FDA; ∴ (E. 6. 1.) $\overline{AF} = \overline{FD}$, a semi-diameter of the circle FEG; ∴ $2\overline{AF}$, which was shewn to be the excess of $\overline{AB} + \overline{AC}$ above $\overline{BC}$, is equal to the diameter of FEG.

PROP. VI.

12. THEOREM. *The straight line bisecting any angle of a triangle, incribed in a given circle, cuts the circumference, in a point which is equi-distant from the extremities of the side opposite to the bisected angle, and from the centre of a circle inscribed in the triangle.*



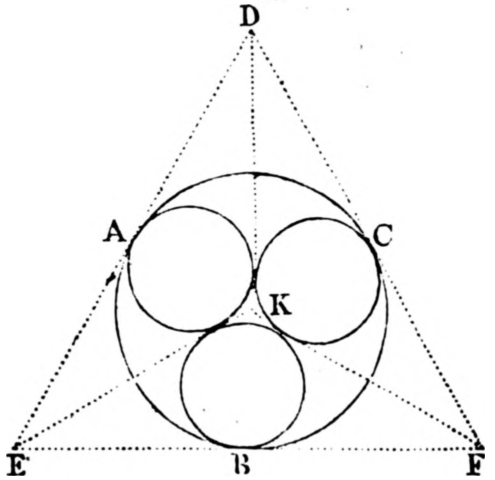
Let KEF be a $\triangle$ inscribed in the circle KEGF, and let $\overline{KG}$, which bisects the $\angle$ EKF, meet the circumference in G: The point G is equi-distant from E and F, and from the centre of the circle inscribed in the $\triangle$ KEF.

For, join G, E, and G, F; draw (E. 9. 1.) $\overline{EI}$ bisecting the $\angle$ KEF; $\therefore$ (E. 4. 4.) I is the centre of the circle inscribed in the $\triangle$ KEF: And since (*hyp.*) the $\angle$ EKG $=$ $\angle$ FKG, $\therefore$ (E. 26. 3.) $\overset{\frown}{GE} = \overset{\frown}{GF}$, and $\therefore$ (E. 29. 3.) $\overline{GE} = \overline{GF}$: Again, because (E. 21. 3.) the $\angle$ GEF $=$ $\angle$ GKF, $\therefore$ (*constr.*) the two $\angle$s GEH, HEI, that is, the $\angle$ GEI, are equal to half of the two $\angle$s EKF, FEK; also the exterior $\angle$ EIG, of the $\triangle$ EIK, is (E. 32. 1.) equal to the two $\angle$s IKE, KEI, that is (*constr.*) to half of the two $\angle$s FKE, KEF; $\therefore$ the $\angle$ EIG $=$ $\angle$ GEI; $\therefore$ (E. 6. 1.) $\overline{GE} = \overline{GI}$; and it has been shewn that $\overline{GE} = \overline{GF}$; $\therefore$ G is equi-distant from E, and F, and from the centre I of the circle inscribed in the $\triangle$ KEF.

PROP. VII.

13. PROBLEM. *In a given circle, to inscribe three equal circles, touching each other and the given circle.*

Let ABC be the given circle: It is required to



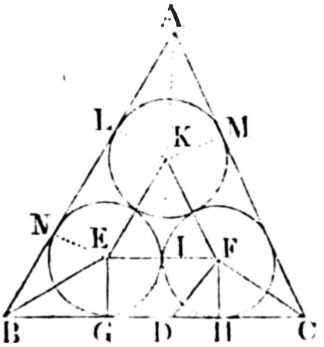
inscribe in it three equal circles, touching each other, and also touching ABC.

About the circle ABC describe (E. 1. 1. and E. 3. 4.) the equilateral △ DEF, the sides of which touch the circle in the points A, B and C: Take the centre K, and draw $\overline{KD}$, $\overline{KE}$ and $\overline{KF}$: It is manifest, from the demonstration of E. 4. 4., and of S. 1. 4. *cor.* 1., that, if a circle be inscribed in each of the equal △s DKE, EKF, FKD, these inscribed circles will be equal, and will touch one another.

PROP. VIII.

14. PROBLEM. *To inscribe three circles in an isosceles triangle, touching each other, and each of them touching two of the three sides of the triangle.*

Let ABC be an isosceles △, having the side



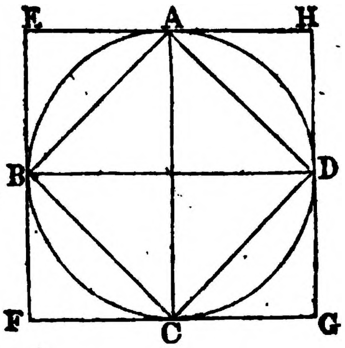
AB equal to the side AC: It is required to inscribe in the △ABC, three circles that touch one another.

Bisect (E. 9. 1.) the vertical ∠ BAC by $\overline{AD}$; ∴ (E. 4. 1.) the two △s ADB, ADC, are right-angled at D; in the △ ADB inscribe (E. 4. 4.) the circle IG; and in the △ ADC inscribe the circle IH: Then, it is manifest, from the demonstration of E. 4. 4, that the two circles IG and IH touch $\overline{AD}$ in the same point I, and ∴ touch one another in that point, and that they are equal to one another; take their centres E and F, and draw $\overline{EF}$, which (E. 12. 3.) passes through I; and since the two circles are equal, EF is bisected in I; join, also, E, G, and F, H; $\overline{EG}$ is (E. 18. 3. and E. 28. 1.) parallel to $\overline{FH}$; and $\overline{EG} = \overline{FH}$, ∴ (E. 33. 1.) $\overline{EF}$ is parallel to $\overline{BC}$, and (E. 29. 1.) the ∠s AIE, AIF, are right ∠s; if, ∴, from the centre E, at the distance EF, a circle be described, cutting AI in K, and if K, F be joined, $\overline{KF}$ (E. 4. 1.) $=$ $\overline{KE}$, and the △ KEF is equilateral; and its vertical ∠ EKF, which (E. 32. 1.) is equal to the ∠ BAC, is bisected by $\overline{AKI}$; ∴ the ∠EKI = ∠BAD; ∴ (E. 28. 1.) KE is parallel to AB; join E, N, and draw (E. 12. 1.) $\overline{KL}$ ⊥ to $\overline{AB}$ and ∴ (E. 28. 1.) parallel to EN; ∴ KLNE is a ▭, and (E. 34. 1.) $\overline{KL} = \overline{EN}$, or $\overline{EI}$, or the half of $\overline{EK}$; and if KM be drawn ⊥ to $\overline{AC}$, it is equal (*constr.* and E. 26. 1.) to $\overline{KL}$. It is evident, ∴, that a circle, LM, described from the centre K, at the distance KL, or KM, will (E. 16. 3. *cor.* and S. 6. 3.) touch $\overline{AB}$ and $\overline{AC}$, and each of the circles GI, and HI: And thus will three circles have been inscribed in the isosceles △ ABC touching one another, and each of them touching two sides of the triangle.

PROP. IX.

15. THEOREM. *The square, inscribed in a circle, is equal to the half of the square upon its diameter.*

Let ABCD be a circle: Inscribe in it (E. 6. 4.) by

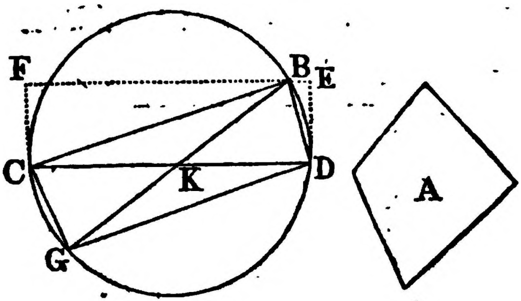


drawing the diameters AC, and BD ⊥ to one another, the square ABCD, and describe about it (E. 7. 4.) the square EFGH: And since (E. 41. 1.) the △ BAD is half of the ▭ EBDH, and the △ BCD is half of the ▭ BFGD, the two △ BAD, BCD are, together, half of the two ▭ EBDH, BFGD; that is, the inscribed square ABCD is half of the circumscribed square EFGH, which is equal to the square upon the diameter, because (E. 34. 1.) its side FG = the diameter BD of the circle.

PROP. X.

16. **PROBLEM.** *In a given circle, to inscribe a rectangle equal to a given rectilineal figure, not exceeding the half of the square upon the diameter.*

Let A be the given rectilineal figure, and BCD the given circle: It is required to inscribe, in the circle BCD, a rectangle equal to the figure A.

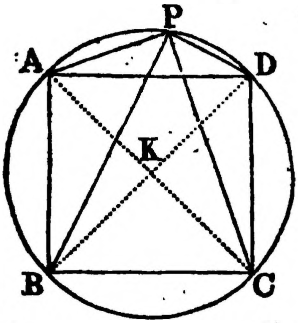


Draw any diameter CD of the given circle; find (S. *55.* 1. *cor.*) a △ equal to A; and to $\overline{CD}$ apply (E. 44. 1.) a ▭ CDEF, equal to that △, and, ∴, equal to the given figure A; let the side EF of the ▭ CDEF cut the circumference of the given circle CBD in B; draw the diameter BKG, and join C, B, and B, D, and C, G and D, G: And, since (E. 31. 3.) each of the ∠∠ CBD, BDG, DGC, GCB, are right ∠∠, ∴ (S. 36. 1.) BDGC is a rectangular ▭; and ∴ (E, 34. 1.) it is double of the △ CBD; also (E. 41. 1.) the ▭ EDCF is double of the △ CBD; ∴ the rectangle CBDG = ▭ EFCD, which has been shewn to be equal to A; ∴ the rectangle CBDG = A.

PROP. XI.

17. THEOREM. *If from any point, in the circumference of a given circle, straight lines be drawn to the four angular points of an inscribed square, the aggregate of the squares of the four lines, so drawn, shall be the double of the square of the diameter.*

Let ABCD be a given circle; inscribe in it (E.



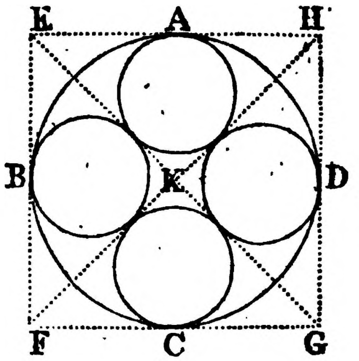
6. 4.) the square BADC, and from any point P, in the circumference, let there be drawn to the angular points A, B, C, D, $\overline{PA}$, $\overline{PB}$, $\overline{PC}$ and $\overline{PD}$: Then $\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 + \overline{PD}^2$ shall be double of the square of the diameter.

For let K be centre of the circle, and $\overline{AKC}$, $\overline{DKB}$ the two diameters perpendicular to one another, by joining the extremities of which (E. 6. 4.) the square was inscribed in the circle: Then since (E. 31. 3.) the ∠∠ APC, BPD are right ∠∠, ∴ (E. 47. 1.) $\overline{PA}^2 + \overline{PC}^2 = \overline{AC}^2$, and $\overline{PB}^2 + \overline{PD}^2 = \overline{DB}^2$ or $\overline{AC}^2$; ∴ $\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 + \overline{PD}^2 = 2\overline{AC}^2$.

PROP. XII.

18. PROBLEM. *In a given circle, to inscribe four circles equal to each other, and in mutual contact with each other and the given circle.*

Let ABCD be the given circle: It is required to inscribe in it four equal circles touching one another, and the circle ABCD.



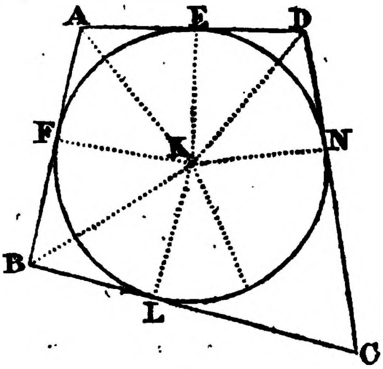
About the circle ABCD describe (E. 7. 4.) the square EFGH, and draw its diagonals EG, HF, which cut one another in the centre K, so that (E. 26. 1.) the four △ EKF, FKG, GKH, HKE, have their sides and ∠ respectively equal to one another: It is manifest, ∴, from the demonstration of E. 4. 4., that if a circle be inscribed in each of the four equal △s, the circles so described, will be equal, and will touch one another, and the given circle ABCD.

19. Cor. In the same manner, four equal circles may be inscribed in a given square, touching each other and the sides of the square.

Prop. XIII.

20. Problem. *To inscribe a circle in a given trapezium, of which two opposite sides are, together, equal to the other two sides taken together.*

Let ABCD be the given trapezium, having the two sides AD and BC equal, together, to the two remaining opposite sides AB and DC: It is re-



quired to inscribe a circle in the trapezium ABCD.

Bisect (E. 9. 1.) each of the $\angle$ BAD, ADC, by $\overline{AK}$ and $\overline{DK}$, which meet in K; from K draw (E. 12. 1.) $\overline{KE} \perp$ to $\overline{AD}$, $\overline{KF} \perp$ to $\overline{AB}$, $\overline{KL} \perp$ to $\overline{BC}$, and $\overline{KN} \perp$ to $\overline{CD}$: Then (*demonstr.* of S. 3. 4.) $\overline{KE}$, $\overline{KF}$, and $\overline{KN}$ are equal to one another; as are, also, $\overline{AF}$ and $\overline{AE}$, and $\overline{DE}$ and $\overline{DN}$; and $\overline{KL}$ is equal to $\overline{KF}$ or $\overline{KN}$: For if $\overline{KL}$ be not equal to $\overline{KF}$ or $\overline{KN}$, it is either greater or less; if it be possible, let $\overline{KL} > \overline{KF}$ or $\overline{KN}$; and join K, A, and K, D, and K, C, and K, B: Then (*constr.* and E. 47. 1.) $\overline{KB}^2 = \overline{KF}^2 + \overline{BF}^2$; and, likewise, $\overline{KB}^2 = \overline{KL}^2 + \overline{BL}^2$; $\therefore \overline{KF}^2 + \overline{BF}^2 = \overline{KL}^2 + \overline{BL}^2$; but $\overline{KL}^2 > \overline{KF}^2$, $\therefore \overline{BL}^2 < \overline{BF}^2$, and $\overline{BL} < \overline{BF}$: In the same manner it may be shewn that $\overline{LC} < \overline{CN}$; $\therefore \overline{BL} + \overline{LC}$, or $\overline{BC}$, $< \overline{BF} + \overline{CN}$; add $\overline{AD}$ to $\overline{BC}$, and $\overline{AF} + \overline{DN}$, which $= \overline{AD}$, to $\overline{BF} + \overline{CN}$; $\therefore \overline{AD} + \overline{BC} <$

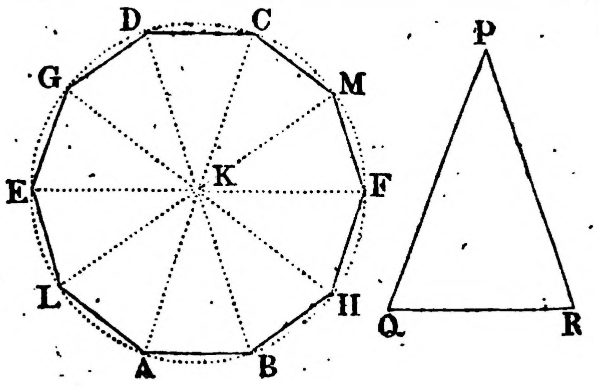
$\overline{AB}+\overline{DC}$, which is contrary to the supposition; $\therefore$ $\overline{KL}$ is not $>$ $\overline{KF}$; and in a similar manner it may be shewn that $\overline{KL}$ is not $<$ $\overline{KF}$; $\therefore$ $\overline{KL}=\overline{KF}$, or $\overline{KN}$, or $\overline{KE}$: From K, $\therefore$, as a centre, at the distance KF, describe a circle EFLN, and it will pass through the points L, G, and E, and (E. 16. 3. *cor.*) will touch $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$, respectively, in the points F, L, N, and E.

21. Cor. If two opposite sides of a trapezium be together equal to the other two sides, taken together, the four straight lines, which bisect the four $\angle\angle$ of the figure, all of them meet in the same point.

Prop. XIV.

22. Problem. *Upon a given finite straight line, to describe an equilateral and equiangular decagon.*

Let AB be the given straight line: It is re-



quired to describe upon it an equilateral and equiangular decagon.

Describe (E. 10. 4.) the △ PQR, having each of the ∠∠ PQR, PRQ, double of the ∠ P; at the points A and B, in $\overline{AB}$, make (E. 23. 1.) the ∠∠ BAK, ABK, each of them equal to the ∠ PQR, or PRQ; ∴ (S. 26. 1.) the ∠ AKB = ∠ QPR, and the ∠∠ KAB, KBA, are each of them the double of the ∠ AKB, which ∠ is, ∴, the fifth part of two right ∠∠; from the centre K, at the distance KA or KB, describe the circle ABCD, cutting $\overline{AK}$ and $\overline{BK}$, produced, in C and D; bisect (E. 9. 1.) the ∠ DKA by $\overline{EKF}$, which (E. 15. 1.) also bisects the ∠ CKB; again bisect the ∠∠ DKE, EKA, by $\overline{GKH}$, $\overline{LKM}$, which also bisect the ∠∠ FKB, CKF; lastly, draw $\overline{BH}$, $\overline{HF}$, $\overline{FM}$, $\overline{MC}$, $\overline{CD}$, $\overline{DG}$, $\overline{GE}$, $\overline{EL}$ and $\overline{LA}$: The ten-sided figure ABHFMCDGEL is an equilateral and equiangular decagon.

For the ∠ AKB has been shewn to be the fifth part of two right ∠∠; ∴ (E. 13. 1.) it is the fifth part of the ∠∠ AKB, AKD; ∴ the ∠ AKD = 4 ∠ AKB; ∴ (*constr.*) the ∠∠ BKA, AKL, LKE, EKG, GKD, and (E. 15. 1.) their vertical ∠∠ are equal to one another; ∴ (E. 26. 3. and E. 29. 3.) the ten-sided figure is equilateral; and since (*constr.* and E. 32. 1.) the isosceles △△, into which it is divided by the straight lines drawn from K to its angular points, have the ∠∠ at their bases all equal, the figure is also equiangular.

23. Cor. 1. It is manifest from this proposition, and from E. 10. 4., that if the semi-diameter of a circle be divided into two parts, so that the rectangle contained by the whole and the lesser part may be equal to the square of the greater part, the greater segment shall be equal to the side of an equilateral and equiangular decagon inscribed in the circle; and thus may such a decagon be inscribed in a given circle.

24. Cor. 2. In the solution of the proposition, is shewn the method of describing, upon a given finite straight line as a base, an isosceles △, having each of the ∠∠ at the base double of the third angle.

25. Cor. 3. The figure ABHFMCDGEL being an equilateral and equiangular decagon, if the points A, H, and H, M, and M, D, and D, E, and E, A, be joined, it may be shewn, from E. 4. 1., that the figure AHMDE is an equilateral and equiangular pentagon.

In the same manner, if an equilateral and equiangular rectilineal figure of any *even* number of sides be given, a similar figure, having half that number of sides, may be constructed: Also, if a circle be described about the given figure, which can always be done by the method used in E. 14. 4., and, each of the equal ∠∠ of the figure having (E. 9. 1.) been bisected, if the points in which the circumference is met by the bisecting lines, and the angular points of the given figure be joined, a figure of twice as many sides as the given figure will have been constructed, which (E. 26. 3., and E. 29. 3.), is equilateral, and, ∴, equiangular: For in the same manner, that an equilateral pentagon, inscribed in a circle, is shewn (E. 11. 2.) to be equiangular, may any other equilateral rectilineal figure, inscribed in a circle, be shewn to be equiangular.

Thus, by the help of E. 9. 1, E. 2. 4, E. 6. 4, E. 11. 4, and E. 16. 4., equilateral and equiangular figures may be inscribed in a given circle, of three, six, twelve, &c., equal sides; of four, eight, sixteen, &c. equal sides; of five, ten, twenty, &c. equal sides; and of fifteen, thirty, sixty, &c. equal sides.

PROP. XV.

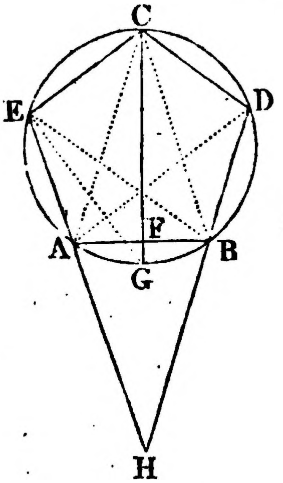
26. PROBLEM. *Upon a given finite straight line, to describe an equilateral and equiangular pentagon.*

If upon the given finite straight line an isosceles △ be described (S. 14. 4. *cor.* 2.) having each of the ∠s at the base double of the third ∠, and if, also, a circle be described (E. 5. 4.) about that △, it will be manifest, that the equilateral and equiangular pentagon inscribed in the circle, according to the method used in E. 11. 4., is the figure which was to be constructed.

Prop. XVI.

27. **Theorem.** *The angle of a regular pentagon exceeds a right angle by one-fifth part of a right angle; and is three times as great as the angle contained by any two sides of the figure, which are not adjacent to each other, produced so as to meet.*

Let ABDCE be the given equilateral and equi-



angular pentagon, and let any two of its sides, as EA, DB, be produced, so as to meet in H: Any of its ∠∠ exceeds a right-angle by one-fifth part of a right ∠, and is three times as great as the ∠ AHB.

About the pentagon ABDCE describe (E. 14. 4.) the circle AECDB; bisect (E. 30. 3.) $\overset{\frown}{AB}$, in G, and join C, A and C, B, and C, G and E, G and E, B: And since (*hyp.* and E. 28. 3.) the circumferences

$\overset{\frown}{CE}$, $\overset{\frown}{EA}$, $\overset{\frown}{CD}$, $\overset{\frown}{DB}$, are equal, $\overset{\frown}{CEG} = \overset{\frown}{CDG}$; $\therefore$ $\overset{\frown}{CEG}$ is the semi-circumference of the circle, and (E. 31. 3.) the $\angle$ CEG is a right $\angle$; also (E. 21. 3.) the $\angle$ AEG $=$ $\angle$ ACG; and it is manifest from the demonstration of E. 11. 4. E. 32. 1., that the $\angle$ ACB is the fifth part of two right $\angle$s, and $\therefore$ that its half, namely the $\angle$ AEG, is the fifth part of a right $\angle$; $\therefore$ the $\angle$ AEG, which is the excess of the $\angle$ CEA above a right $\angle$, is the fifth part of a right $\angle$.

Again, the two opposite $\angle$s AEC, CBA, of the trapezium AECB are (E. 22. 3.) together equal to two right $\angle$s; and (E. 27. 3.) the $\angle$ CBA $=$ $\angle$ BCE; $\therefore$ the $\angle$ AEC $+$ $\angle$ ECB $=$ two right $\angle$s, and $\therefore$ (E. 28. 1.) $\overline{CB}$ is parallel to $\overline{EA}$ or $\overline{EH}$; $\therefore$ (E. 29. 1.) the $\angle$ EHD $=$ $\angle$ CBD, which, since (E. 27. 3.) the three $\angle$s CBD, CBE, and EBA, are equal to one another, is a third part of the $\angle$ ABD of the pentagon ABDCE.

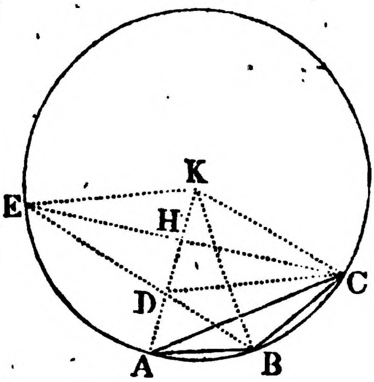
28. Cor. It is manifest from the demonstration, that the straight line joining the extremities of the first and second side of an equilateral and equiangular pentagon is parallel to the fourth side of the figure; the sides being taken in order from any one of them assumed as the first.

Prop. XVII.

29. Theorem. *The square of the side of a regular pentagon, inscribed in a given circle, is equal to*

the square of the side of a regular decagon, together with the square of the side of the regular hexagon, both inscribed in that given circle.

Let AECB be the given circle, of which K is



the centre, and $\overline{AB}$ the side of a regular decagon inscribed (S. 14. 4. *cor.* 1.) in it: Place, in the circle, $\overline{BC}=\overline{AB}$, and join A, C; ∴ (S. 14. 4. *cor.* 3.) $\overline{AC}$ is the side of a regular pentagon inscribed in the circle, and if $\overline{KA}$, $\overline{KB}$, and $\overline{KC}$ be drawn, any one of these lines, as KA, is (E. 15. 4.) the side of a regular hexagon inscribed in the circle: Then $\overline{AC}^2=\overline{KA}^2+\overline{AB}^2$.

For, in $\overline{KA}$ take $\overline{KD}=\overline{AB}$, draw $\overline{BD}$ and produce it to meet the circumference in E; also, draw $\overline{KE}$, $\overline{KC}$, $\overline{CE}$ and $\overline{CD}$: Then, it is manifest from S. 14. 4. *cor.* 1. and E. 10. 4., that the $\angle$ KBD $=$ $\angle$ AKB; but (*constr.* and E. 8. 1.) the $\angle$ AKB $=$ $\angle$ BKC; ∴ the $\angle$ DBK or EBK $=$ $\angle$ BKC; ∴ (E. 27. 1.) $\overline{ED}$ is parallel to $\overline{KC}$:

Again, (E. 20. 3.) the ∠ EKA or EKD = 2 ∠ EBA; but, (E. 32. 1.) the exterior ∠ EDK is equal to the ∠ DKB + ∠ KBD, that is, (*constr.*) to twice the ∠ KBE, or to twice the ∠ EBA; ∴ the ∠ EKD = ∠ EDK, ∴ $\overline{ED}=\overline{EK}$; and (E. 15. def. 1.) $\overline{EK}=\overline{KC}$; ∴ $\overline{ED}=\overline{KC}$, and it has been shewn that $\overline{ED}$ is parallel to $\overline{KC}$; ∴ (E. 33. 1.) $\overline{EK}$ is equal and parallel to $\overline{DC}$, and the figure EKCD is a rhombus; ∴ (S. 45. 1.) $\overline{KD}$ is bisected at right ∠∠ in H, by $\overline{CE}$: And since $\overline{KD}$ is bisected in H and produced to A,

∴ (E. 6. 2.)

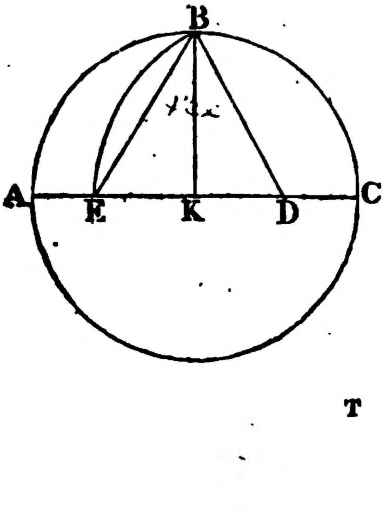
$$\overline{KA}\times\overline{AD}+\overline{DH}^2=\overline{AH}^2$$

$$\therefore \overline{KA}\times\overline{AD}+\overline{DH}^2+\overline{HC}^2=\overline{AH}^2+\overline{HC}^2;$$

but (*constr.* and S. 14. 4. *cor.* 1.) $\overline{KA}\times\overline{AD}=\overline{AB}^2$; and (E. 47. 1.) $\overline{DH}^2+\overline{HC}^2=\overline{DC}^2$, or $\overline{KC}^2$, or $\overline{KA}^2$; and $\overline{AH}^2+\overline{C}^2=\overline{AC}^2$;

$$\therefore \overline{AC}^2=\overline{KA}^2+\overline{AB}^2.$$

30. **Cor.** Hence, if ABC be a given circle,



and $\overline{AC}$, $\overline{BK}$ two diameters drawn (E. 11. 1.) at right ∠ to one another, if $\overline{KC}$ be (E. 10. 1.) bisected in D, and if from $\overline{DA}$ there be cut off $\overline{DE}=\overline{DB}$, $\overline{EK}$ is the side of a regular decagon inscribed in the circle ABC, $\overline{KB}$ is the side of a regular hexagon inscribed in it, and $\overline{EB}$ is the side of a regular pentagon inscribed in it.

For (E. 11. 2.) $\overline{EK}$ is equal to that part of $\overline{KB}$, the square of which equals the rectangle contained by $\overline{KB}$ and the remaining part of $\overline{KB}$; ∴ (S. 14. 4. *cor.* 1.) $\overline{EK}$ is the side of a regular decagon; and $\overline{KB}$ (E. 15. 4.) is the side of the regular hexagon, inscribed in the circle ABC; since, ∴, (*constr.* and E. 47. 1.) $\overline{EB}^2=\overline{EK}^2+\overline{KD}^2$, $\overline{EB}$ is (S. 17. 4.) the side of a regular pentagon inscribed in the circle ABC.*

PROP. XVIII.

31. PROBLEM. *Upon a given finite straight line, to describe an equilateral and equiangular hexagon.*

Upon the given straight line, as a base, describe

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* This corollary furnishes the best practical method of determining the sides of a regular pentagon, and of a regular de-

(E. 1. 1.) an equilateral △; from its vertex as a centre, at the distance of either of its sides, describe a circle: Then, if a regular hexagon be inscribed in the circle by the method used in E. 15. 4., taking either extremity of the base of the equilateral △, for the centre of the circle to be next described, it is manifest that the given straight line will be one of its sides.

PROP. XIX.

32. PROBLEM. *A circle being given, to describe six other circles, each of them equal to it, and in contact with each other and with the given circle.*

Let IGH be the given circle: It is required to describe six other circles, equal each of them to the circle IGH, and touching that circle and each other.

Find (E. 1. 3.) the centre K, of the circle IGH; take any of its semi-diameters, as KG; produce $\overline{KG}$ to A, and make $\overline{GA}=\overline{GK}$; from the centre K, at the distance KA, describe the circle ABCD; and in the circle ABCD inscribe (E. 15. 4.) the equilateral and equiangular hexagon ABFCDE; bisect (E. 10. 1.) the side AB of the hexagon in

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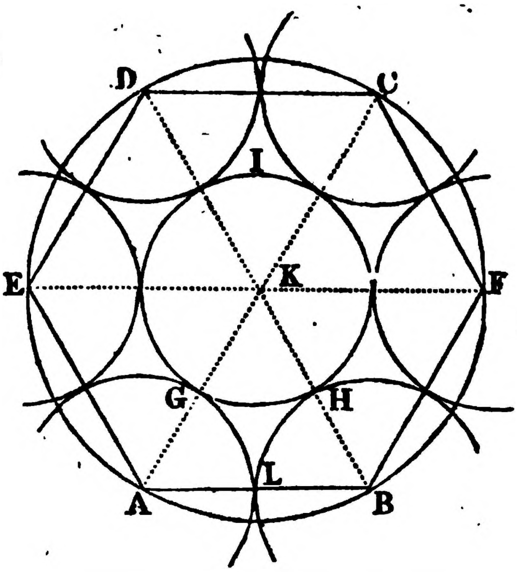
cagon, to be inscribed in a given circle; and thus makes it easier to describe a regular pentagon, or a regular decagon, on a given finite straight line.



L: Then since (E. 15. 4.) $\overline{AB} = \overline{KA}$ or $\overline{KB}$, and (*constr.*) $\overline{AG}$ and $\overline{BH}$ are, each of them, the half of $\overline{KA}$ or $\overline{KB}$; ∴ $\overline{AG}$ and $\overline{BH}$ are each of them equal to $\overline{AL}$ or $\overline{BL}$: If, ∴, from the centres A and B, at the distance $\overline{AG}$, or $\overline{BH}$, two circles be described, they will be equal to one another and to the given circle, and they will touch (S. 6. 3.) the given circle in G and H, and will, also, touch one another in L. In the same manner, from the points E, D, C, F, as centres, may four other circles be described, each equal to the given circle and in contact with it, and touching also each other.

33. Cor. No more than six circles can be described touching one another and a given circle, and each of them equal to the given circle.

PROP. XX.

34. **PROBLEM.** *In a given circle to inscribe six circles equal to one another, touching, each of them, the given circle, and touching, also, one another.*

Inscribe (E. 15. 4.) an equilateral and equiangular hexagon in the given circle, and through the points, in which are its ∠, draw, (E. 17. 3.) straight lines touching the circle; and it may be shewn, by the method used in E. 12. 4., that the figure contained by these tangents is an equilateral and equiangular hexagon; from the centre of the circle draw straight lines to the several ∠ of the circumscribed hexagon, thus dividing it into six equal equilateral △; and if (E. 4. 4.) a circle be inscribed in each of these △, it will be manifest from the demonstration of E. 4. 4., that the circles, so inscribed, will be equal, and that they will touch one another in common points of the sides of the △, and will, also, touch the given circle, each of them in one of the points of contact of the circumscribed hexagon.

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BOOK V.

PROP. I.

1. **THEOREM.** *If the first of four proportional magnitudes be greater than the second, the third is also greater than the fourth; if equal, equal; and if less, less.*

Let $A : B :: C : D$; and first, let $A > B$; then $C > D$.

Take the doubles of the four magnitudes; and since (*hyp.*) $A > B$, twice A > twice B; ∴ (*hyp.* and 5 def. 5.) twice C > twice D; ∴ $C > D$.

In like manner it can be shewn, if $A = B$, that $C = D$; and if $A < B$, that $C < D$.

PROP. II.

2. **THEOREM.** *If four magnitudes are propor-*

tionals, they are proportionals also when taken inversely.

Let A:B::C:D; then B:A::D:C.

Let there be taken of A and C any equi-multiples pA, pC, and of B and D any equi-multiples qB, qD: Then (*hyp.*) A : B :: C : D, ∴ (5. def. 5.) if pA > qB, pC > qD; if pA = qB, pC = qD; if pA < qB, pC < qD; ∴ accordingly as qB is greater than, equal to, or less than pA, qD is greater than, equal to, or less than pC;

∴ (5. def. 5.) B : A :: D : C.

PROP. III.

3. THEOREM. *If the first of four magnitudes be the same multiple of the second, or the same part of it, that the third is of the fourth, the first is to the second, as the third is to the fourth.*

First, let A = pB, and C = pD; then A : B :: C : D.

Let there be taken of A and C any equi-multiples qA, qC, and of B and D any equi-multiples rB, rD. And since A = pB, accordingly as qA >, =, or < rB, will q times pB be >, =, or < rB, *i. e.* q times p will be >, =, or < r; and ∴ q times pD will also be >, =, or < rD; *i. e.* (*hyp.*) qC will be >, =, or < rD; ∴ (5. def. 5.)

A : B :: C : D.

Secondly, let $pA = B$, and $pC = D$; then, also,

$$A : B :: C : D.$$

For in that case, as hath been shewn,

$$B : A :: D : C;$$

$$\therefore \text{(S. 2. 5.)}\ A : B :: C : D.$$

PROP. IV.

4. THEOREM. *If the first of four proportional magnitudes be a multiple, or a part, of the second, the third is the same multiple, or the same part, of the fourth.*

If $A : B :: C : D$, and if $A = pB$, then $C = pD$.

For (*hyp.* and S. 3. 5.) $A : B :: pD : D$;

and (*hyp.*) $A : B :: C : D$;

$\therefore$ (E. 11. 5.) $C : D :: pD : D$;

$\therefore$ (E. 9. 5.) $C = pD$.

Again, if $A : B :: C : D$, and if $pA = B$, then $pC = D$.

For (*hyp.*) $A : B :: C : D$;

$\therefore$ (S. 2. 5.) $B : A :: D : C$;

and (*hyp.*) $B = pA$; $\therefore$, as in the former case, $D = pC$; *i. e.* C is the same part of D, that A is of B.

PROP. V.

5. **THEOREM.** *If any number of equal ratios be each greater than a given ratio, the ratio of the sum of their antecedents to the sum of their consequents, shall be greater than that given ratio.*

Let the ratios (A : B), (C : D), (E : F), &c. be equal to one another, and let each of them be greater than the ratio (P : Q); then (A + C + E : B+D+F) > (P : Q.)

For (E. 12. 5.) A+C+E : B+D+F : : A : B;
and (*hyp.*) (A : B) > (P : Q);
∴ (A+C+E : B+D+F) > P : Q.

PROP. VI.

6. **THEOREM.** *If the first of four magnitudes have a greater ratio to the second than the third has to the fourth, the second shall have to the first a less ratio than the fourth has to the third.*

If (A : B) > (C : D), then is (B : A) < (D : C).
For, let E be a magnitude such that
(E : B) : : (C : D);

and since (*hyp.*)

$(A : B) > (C : D)$ ∴, $(A : B) > (E : B)$;

∴ (E. 10. 5.) $E < A$;

∴ (E. 8. 5.) $(B : E) > (B : A)$;

But (*hyp.* and S. 2. 5.) $(D : C) :: (B : E)$;

∴ (E. 13. 5.) $(D : C) > (B : A)$:

Or, $(B : A) < (D : C)$.

PROP. VII.

7. THEOREM. *If the first of four magnitudes, of the same kind, have a greater ratio to the second than the third has to the fourth, the first shall have to the third a greater ratio than the second has to the fourth.*

If $(A : B)$ be greater than $(C : D)$, then is $(A : C) > (B : D)$.

For, let E be a magnitude such that

$(E : B) :: (C : D)$:

∴ (*hyp.* and E. 10. 5.) $A > E$

∴ (E. 8. 5.) $(A : C) > (E : C)$;

But (E. 16. 5. and *hyp.*) $(E : C) :: (B : D)$

∴ $(A : C) > (B : D)$.

PROP. VIII.

8. THEOREM. *If four magnitudes of the same kind be proportionals, and if the first of them be the greatest, the fourth shall be the least; but if the first of them be the least, the fourth shall be the greatest.*

Let A, B, C, D, be four magnitudes of the same kind, which are proportionals; and, first, let A be the greatest; then D shall be the least of them.

For, since (*hyp.*) $A > C$, ∴ (E. 14. 5.) $B > D$;

Again, since (*hyp.*) $A : B :: C : D$,

∴ (E. 16. 5.) $A : C :: B : D$:

But (*hyp.*) $A > B$; ∴ (E. 14. 5.) $C > D$: And it has been shewn that $B > D$; ∴ D is in this case the least of the four proportionals. And, if A be the least of the four proportionals, it may, in like manner, be proved that D will be the greatest of them.

9. COR. If four magnitudes, of the same kind, be proportionals, the difference between the two extremes is greater than the difference between the two means.

PROP. IX.

10. THEOREM. *If the first, together with the second, of four magnitudes, have a greater ratio*

to the second, than the third, together with the fourth, has to the fourth, the first shall have a greater ratio to the second than the third has to the fourth.

If $(A + B : B)$ be greater than $(C + D : D)$, then is $(A : B) > (C : D)$

For, let E be a magnitude such that $(E+B : B) :: (C+D : D)$;

$$\therefore \text{(E. 10. 5.)}\ A+B > E+B;$$

$$\therefore A > E;$$

$$\therefore \text{(E. 8. 5.)}\ (A : B) > (E : B):$$

But (*hyp.* and E. 17. 5.) $(E : B) = (C : D)$;

$$\therefore (A : B) > (C : D).$$

PROP. X.

11. THEOREM. *If the first of four magnitudes have a greater ratio to the second than the third has to the fourth, the first, together with the second, shall have to the second, a greater ratio than the third, together with the fourth, has to the fourth.*

If $(A : B)$ be greater than $(C : D)$, then is $(A + B : B) > (C+D : D)$.

For, let E be a magnitude such that $(E : B) :: (C : D)$;

$\therefore$ (E. 10. 5.) $A > E$;

$\therefore A+B > E+B$;

$\therefore$ (E. 8. 5.) $(A+B : B) > (E+B : B)$:

But (E. 18. 5. and *hyp.*) $(E+B : B.) :: (C+D : D)$;

$\therefore (A+B : B) > (C+D : D)$.

PROP. XI.

12. **THEOREM.** *If the first term of a ratio be less than the second, the ratio shall be increased by adding the same quantity to both terms; but if the first term be greater than the second, the ratio shall be diminished by adding the same quantity to both.*

Let A be less than B, and let C be any other magnitude:

Then is $(A+C : B+C) > (A : B)$.

For, (E. 8. 5. and *hyp.*), $(C : A) > (C : B)$;

$\therefore$ (S. 10. 5.), $(A+C : A) > (B+C : B)$;

$\therefore$ (S. 7. 5.) $(A+C : B+C) > (A : B)$.

And, if A be greater than B, it may, in the same manner, be shewn that $(A+C : B+C) < (A : B.)$

PROP. XII.

13. **THEOREM.** *If the first of four magnitudes, of the same kind, have a greater ratio to the se-*

cond than the third has to the fourth, the first, together with the third, shall have to the second, together with the fourth, a greater ratio than the third has to the fourth, and a less ratio than the first has to the second.

If (A:B) be greater than (C:D), then is (A+C:B+D) > (C:D); and (A+C:B+D) < (A:B).

For, (S. 7. 5. and *hyp.*) (A:C) > (B:D);

∴ (S. 10. 5.), (A+C:C) > (B+D:D);

∴ (S. 7. 5.), (A+C:B+D) > (C:D):

Again, since (*hyp.* and S. 6. 5.), (B:A) < (D:C), or (D:C) > (B:A), it may be shewn, in the same manner, that

(A+C:B+D) < A:B.

PROP. XIII.

14. THEOREM. *If the first, together with the second, have to the second, a greater ratio than the third, together with the fourth, has to the fourth, then shall the first, together with the second, have to the first, a less ratio than the third, together with the fourth, has to the third.*

If (A+B:B) be greater than (C+D:D), then is (A+B:A) < (C+D:C).

For (S. 9. 5. and *hyp.*) (A:B) > (C:D);

∴ (S. 6. 5.) (B:A) < (C:D);

∴ (S. 10. 5.) (A+B:A) < (C+D:C).

PROP. XIV.

15. **THEOREM.** *If the first, together with the second, have to the third, together with the fourth, a greater ratio than the first has to the third; then shall the second have to the fourth a greater ratio, than the first, together with the second, has to the third, together with the fourth.*

If $(A+B:C+D)$ be greater than $(A:C)$, then is $(B:D)>(A+B:C+D)$.

For (*hyp.* and S. 7. 5.)

$$(A+B:A)>(C+D:C);$$

∴ (S. 13. 5.) $(A+B:B)<(C+D:C)$;

∴ (S. 7. 5.) $(A+B:C+D)<(B:D)$;

Or, $(B:D)>(A+B:C+D)$.

PROP. XV.

16. **THEOREM.** *If any number of magnitudes be continual proportionals, their differences shall also, be proportionals.*

Let $A:B::B:C::C:D$, &c., then shall $A-B:B-C::B-C:C-D$, and so on.

For (*hyp.* and E. 19. 5.)

$$A:B::A-B:B-C;$$

$$\text{and } B:C::B-C:C-D;$$

$\therefore$ (*hyp.* and E. 11. 5. *cor.*)

$$A-B:B-C::B-C:C-D.$$

17. Cor. From the demonstration it is manifest, that, if three magnitudes, A, B, C, are proportionals, the excess of the greatest, A, above the mean B, is greater than the excess of the mean B above the least, C.

For it has been shewn that $A:B::A-B:B-C$;
And (*hyp.*) $A>B$; $\therefore$ (S. 1. 5.) $A-B>B-C$.

Prop. XVI.

18. Theorem. *If four magnitudes be proportionals, they are also proportionals by conversion: that is, the first is to its excess above the second, as the third to its excess above the fourth.*

Let $A+B:B::C+D:D$; then $A+B:A::C+D:C$.

For (*dividendo*) $A:B::C:D$;
$\therefore$ (*invertendo*) $B:A::D:C$;
$\therefore$ (*componendo*) $A+B:A::C+D:C$.

PROP. XVII.

19. THEOREM. *If there be three magnitudes, and other three, and if the first have a greater ratio to the second, in the former set, than the first has to the second, in the latter; and if, also, the second have to the third, in the former set, a greater ratio than the second has to the third, in the latter; then shall the first have a greater ratio to the third, in the former set, than the first has to the third, in the latter.*

Let A, B, C, be three magnitudes, and D, E, F, three other magnitudes: If (A:B) be greater than (D:E), and (B:C) greater than (E:F), then is (A:C) > (D:F).

For let G be a magnitude such that (G:C) :: (E:F);

∴ (*hyp.* and E. 10. 5.) B > G;

∴ (E. 8. 5.), (A:G) > (A:B).

Again, let H be a magnitude such that (H:G) :: (D:E);

∴ (*hyp.* and E. 13. 5.) (H:G) < (A:B):

Much more then is (H:G) < (A:G);

∴ (E. 10. 5.), A > H;

∴ (E. 8. 5), (A:C) > (H:C):

But (*hyp.* and E. 22. 5.), (H:C) :: (D:F);

∴ (A:C) > (D:F).

PROP. XVIII.

20. THEOREM. *If there be three magnitudes, and other three, and if the first have to the second, in the former set, a greater ratio than the second has to the third, in the latter; and if, also, the second have to the third, in the former set, a greater ratio than the first has to the second, in the latter; then shall the first have to the third, in the former set, a greater ratio, than the first has to the third, in the latter.*

Let A, B, C, be three magnitudes, and D, E, F, three other magnitudes: If $(A:B)$ be greater than $(E:F)$, and $(B:C)$ greater than $(D:E)$, then is $(A:C)>(D:F)$.

For let G be a magnitude such that $(G:C)$ :: $(D:E)$;

∴ (*hyp.* and E. 10. 5.) $B>G$;

∴ (E. 8. 5.), $(A:G)>(A:B)$:

Again, let H be a magnitude such that $(H:G)$ :: $(E:F)$;

∴ (*hyp.* and E. 13. 5.), $(H:G)<(A:G)$;

∴ (E. 10. 5.), $A>H$;

∴ (E. 8. 5.), $(A:C)>(H:C)$;

But (*hyp.* and E. 23. 5.) $(H:C)$ :: $(D:F)$;

∴ $(A:C)>(D:F)$.

PROP. XIX.

21. THEOREM. *If three magnitudes be proportionals, the two extremes are, together, greater than the double of the mean.*

Let A, B, C, be three magnitudes which are proportionals: Then $A+C>2B$.

For (*hyp.* and E. 6. def. B. 5.), $(A:B)::(B:C)$;

$\therefore$ (E. 25. 5.) $A+C>B+B$

i. e. $A+C>2B$.

22. COR. An arithmetic mean proportional, between two given magnitudes, is greater than a geometric mean proportional between the same two magnitudes.

PROP. XX.

23. THEOREM. *If there be two sets of magnitudes, the one geometric, and the other arithmetic, proportionals, and if the two first magnitudes be the same in both, any other magnitude in the former set, shall be greater than the corresponding magnitude in the latter.*

Let the magnitudes A, B, C, D, E, &c. be geometric proportionals, and let the magnitudes A, B, c, d, e, &c. be arithmetic proportionals; then is $C > c$, $D > d$, $E > e$, and so on.

For, first, let A be the least magnitude, in each series;

$$\therefore \text{(S. 15. 5. } \textit{cor.}\text{)}\ C-B > B-A:$$

But, from the property of arithmetic proportion,

$$B-A = c-B;$$
$$\therefore C-B > c-B;$$
$$\therefore C > c.$$

Again, (S. 15. 5. *cor.*) $D-C > C-B$, and as hath been shewn, $C-B > c-B$ or $d-c$; $\therefore D-C > d-c$; and $C > c$; $\therefore D > d$. In the same manner it may be shewn that $E > c$, and so on.

Secondly, let A be the greatest magnitude in each series:

Then (S. 15. 5. *cor.*) $A-B > B-C$;

But, from the property of arithmetic proportion,

$$A-B = B-c;$$
$$\therefore B-c > B-C$$
$$\therefore C > c.$$

Again, (S. 15. 5. *cor.*) $B-C > C-D$; and it has been shewn that $C > c$; much more then is $B-c > C-D$:

$$\text{But } B-c = c-d;$$
$$\therefore c-d > C-D;$$
$$\therefore D > d:$$

And, in the same manner, it may be shewn that, in this case, also, $E > e$, and so on.

24. Cor. The two first magnitudes, in both

the sets, being the same, if the second of the geometric proportionals be greater than the second of the arithmetic proportionals, then, much more, will every other magnitude, in the former set, be greater than the corresponding magnitude in the latter.

PROP. XXI.

25. THEOREM. *If there be two series of magnitudes, the one arithmetically proportional, the other geometrically proportional, but each having the same magnitude for its first term, and if the last term of the arithmetic series be not less than the last term of the geometric series, any other term of the former series shall be greater than the corresponding term in the latter.*

Let the magnitudes A, B, C, D, E, &c. Q, be geometric proportionals; and let the magnitudes A, b, c, d, e, &c. q, be arithmetic proportionals; then if q be not less than Q, $b > B$, $c > C$, $d > D$, and so on.

For (S. 20. 5. and *cor.*) if B be equal to b, or greater than b, $Q > q$; which is contrary to the hypothesis; $\therefore b > B$:

Again, in the two series B, C, D, &c. Q, b, c, d, &c. q, let b, which has been shewn to be greater than B, be supposed to become equal to B, and q to remain of a magnitude not less than Q; then it is manifest, from the nature of arithmetic proportion, that the intermediate terms C, d, &c. must each, also, become less than they are in the given arithmetic series; and yet, as hath been shewn, the second of them will be greater than C; much more, then, is the term c, in that given series, greater than C: And, in the same manner, it may be proved that $d > D$; and so on.

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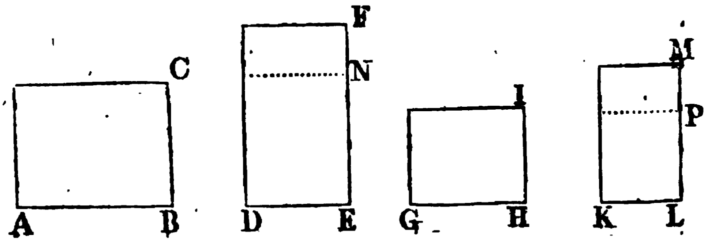
BOOK VI.

PROP. I.

1. **THEOREM.** *If the bases of four rectangles be proportionals, and their altitudes be also proportionals, the rectangles themselves shall likewise be proportionals.*

Let the four rectangles AC, DF, GI, KM, have



their bases AB, DE, GH, KL proportionals, and let their altitudes BC, EF, HI, LM, also, be proportionals: Then AC : DF :: GI : KM.

For, in $\overline{EF}$ and $\overline{LM}$, produced if necessary, take $\overline{EN}=\overline{BC}$, and $\overline{LP}=\overline{HI}$, and complete the rectangles DN and KP.

Then since (*hyp.*) $\overline{AB} : \overline{DE} :: \overline{GH} : \overline{KL}$,
∴ (*constr.* E. 1. 6. and E. 11. 5.)
AC : DN :: GI : KP:
Also, (*hyp.* and *constr.*) $\overline{EN} : \overline{EF} :: \overline{LP} : \overline{LM}$.
∴ (E. 1.6. and E. 11. 5.) DN : DF :: KP : KM.
∴ (E. 22. 5.) AC : DF :: GI : KM.

2. Cor. 1. If four straight lines be proportionals, their squares shall also be proportionals.

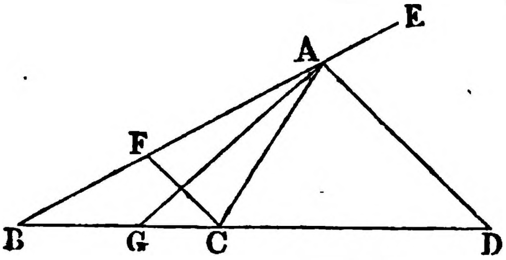
3. Cor. 2. Conversely, if four squares be proportionals, their sides shall likewise be proportionals.

Prop. II.

4. Theorem. *If the outward angle of a triangle, made by producing one of its sides, be divided into two equal angles, by a straight line which also cuts the base produced, the segments between the dividing line and the extremities of the base have the same ratio, which the other sides of the triangle have to one another: And if the segments of the base, produced, have the same ratio which the other sides of the triangle have, the straight line, drawn from the vertex to the point of section,*

divides the outward angle of the triangle into two equal angles.

First, let the outward ∠CAE, of any △ABC,



be divided into two equal ∠∠, by $\overline{AD}$, which cuts the base BC, produced, in D: Then BD : DC :: BA : AC.

Through C draw (E. 31. 1.) $\overline{CF}$ parallel to AD; ∴ (E. 29. 1.) the ∠ACF = ∠CAD; but (*hyp.*) the ∠CAD = ∠DAE; ∴ the ∠ACF = ∠DAE. Again (*constr.* and E. 29. 1.) the ∠DAE = ∠CFA; and it has been shewn that the ∠ACF = ∠DAE; ∴ the ∠ACF = ∠CFA, and (E. 6. 1.) AF = AC. Also (*constr.* and E. 2. 6.) BD : DC :: BA : AF; *i. e.* BD : DC :: BA : AC, because AF = AC.

Secondly, let BD : DC :: BA : AC, and let $\overline{AD}$ be drawn; then the ∠CAD = ∠DAE.

The same construction having been made, since
(*hyp.*) BD : DC :: BA : AC,
and (*constr.* and E. 2. 6.) BD : DC :: BA : AF,
∴ (E. 11. 5.) BA : AC :: BA : AF;
∴ (E. 9. 5.) AC = AF.

Wherefore (E. 5. 1.) the $\angle AFC = \angle ACF$; but (*constr.* and E. 29. 1.) the $\angle EAD = \angle AFC$, and the $\angle CAD = \angle ACF$; $\therefore$ the $\angle EAD = \angle CAD$.

5. Cor. 1. Hence a given finite straight line may be cut in harmonic proportion.

For let BD be the given finite straight line: Take any point A, out of BD, and through A draw $\overline{BAE}$; join A, D; at the point A, in DA, make (E. 23. 1.) the $\angle DAC = \angle DAE$, and bisect (E. 9. 1.) the $\angle BAC$ by AG: Then is BD cut harmonically in the points G and C.

For (*constr.* and E. 3. 6.) $BG : GC :: BA : AC$;
And (*constr.* and S. 2. 6.) $BA : AC :: BD : DC$;
$\therefore$ (E. 11. 5.) $BG : GC :: BD : DC$;
$\therefore$ (E. 16. 5.) $BG : BD :: GC : DC$;
that is, $BG : BD :: BC - BG : BD - BC$;
which is the property of harmonic proportion.

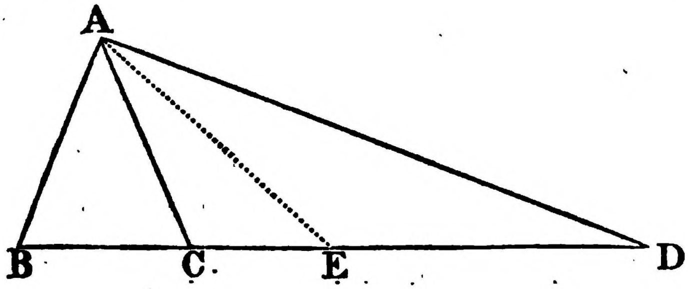
6. Cor. 2. If any straight line be drawn between BE and BD, it may, in the same manner, be shewn to be cut harmonically by the straight lines AG, AC, and AD.

Prop. III.

7. Theorem. *Either of the equal sides of an isosceles triangle, is a mean proportional between the base, and the half of the segment of the base, produced if necessary, which is cut off by a*

straight line drawn from the vertex at right angles to the equal side.

Let ABC be an isosceles △, having the side



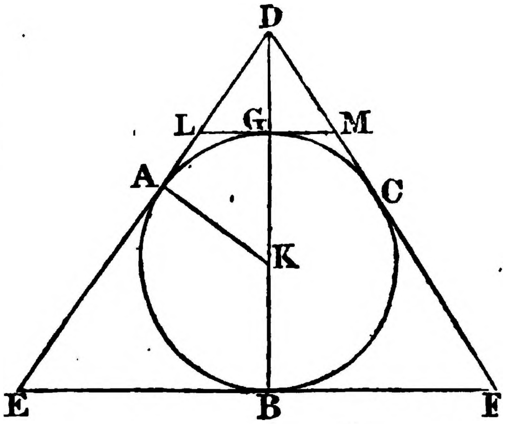
AB = AC, and let $\overline{AD}$, drawn ⊥ to AB, meet BC, produced, if it be necessary, in D; also, let BD be bisected in E: Then BC : AB :: AB : BE.

For draw $\overline{AE}$; and (S. 29. 1.) EA = EB; ∴ (E. 5. 1.) the ∠ EAB = ∠ ABE = ∠ ACB; ∴ (E. 32. 1.) the ∠ AEB = ∠ BAC; ∴ (E. 4. 6.) CB : BA :: BA : BE.

PROP. IV.

8. THEOREM. *The diameter of a circle is a mean proportional between the sides of an equilateral triangle and hexagon described about the circle.*

Let DEF be an equilateral △, described about the circle ABC, of which the centre is K; let the sides of the △ DEF touch the circle in the points



A, B, C; let D, B be joined, cutting the circumference in G, and let $\overline{LM}$ be drawn touching the circle in G; so that (S. 1. 4. *cor.* 2.) $\overline{LM}$ *is the* side of a regular hexagon described about the circle ABC, and GB passes through the centre K; Then, $\overline{DE} : \overline{GB} :: \overline{GB} : \overline{LM}$.

For join A, K; ∴ (E. 18. 1.) the ∠∠ DAK, DGL, are right ∠∠, and the ∠ ADK is common to the two △△ DAK, DGL, which (S. 26. 1.) are, ∴, equiangular;

$$\therefore \text{(E. 4. 6.)}\ \overline{DA} : \overline{AK} :: \overline{DG} : \overline{GL} :$$

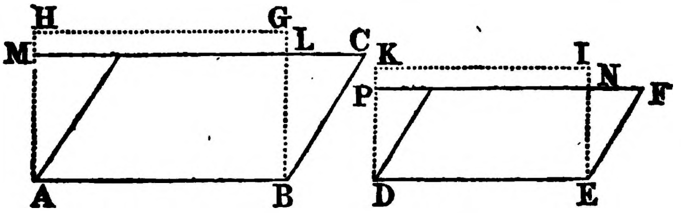
But (S. 1. 4. and *cor.* 1. 2.) $\overline{DE}$ is double of DA; the diameter GB is double of $\overline{AK}$, or of $\overline{DG}$, which (S. 1. 4. *cor.* 1.) is equal to AK; and $\overline{LM}$ is double of $\overline{LG}$:

$$\therefore \text{(E. 15. 5.)}\ \overline{DE} : \overline{GB} :: \overline{GB} : \overline{LM}.$$

PROP. V.

9. **THEOREM.** *Equiangular parallelograms have to one another the same ratio as the rectangles contained by the sides about equal angles in each.*

Let AC, DF, be two equiangular parallelograms,



having the ∠ ABC = ∠ DEF: Then AC : DF :: $\overline{AB} \times \overline{BC} : \overline{DE} \times \overline{EF}$.

For draw (E. 11. 1.) $\overline{BG}$ and $\overline{EI}$ ⊥ to $\overline{AB}$ and $\overline{DE}$, respectively; make $\overline{BG} = \overline{BC}$, and $\overline{EI} = EF$; and complete the rectangles ABGH and DEIK; and produce the sides of the given ▱, that are opposite to AB and DE, to meet $\overline{AH}$ and $\overline{DK}$, in M and P, respectively.

And, since (*hyp.*) the ∠ ABC = ∠ DEF, and (*constr.*) the ∠ ABL = ∠ DEN, ∴ the ∠ LBC = ∠ NEF; also (*hyp.*) the ∠ LCB = ∠ NFE; ∴ (S. 26. 1.) the two △ LBC, NEF, are equiangular:

∴ (E. 4. 1.) BL : BC or BG : : EN : EF or EI:
But (E. 1. 6.) BL : BG : : AL : AG;
Also, EN : EI : : DN : DI;
∴ (E. 11. 1.) AL : AG : : DN : DI;
∴ (E. 16. 5.) AL : DN : : AG : DI :

But (E. 35. 1.) ▭ AL = ▭ AC; and ▭ DN = ▭ DF:

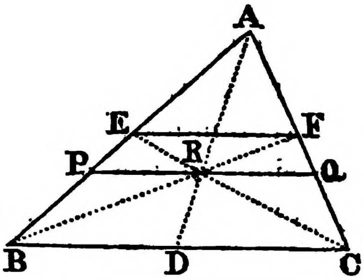
∴ AC : DF : : AG or $\overline{AB} \times \overline{BC}$: DI or $\overline{DE} \times \overline{EF}$.

10. Cor. Triangles, having equal vertical angles, are to one another as the rectangles contained by the sides about those equal angles.

Prop. VI.

11. Theorem. *The straight lines, drawn from the bisections of the three sides of a triangle to the opposite angles, meet in the same point.*

Let the sides AB, AC, of the △ ABC, be bisected (E. 10. 1.) in E and F; and let $\overline{BF}$ and $\overline{CE}$ cut one another in the point R: The straight line which is drawn from A, to the bisection of $\overline{BC}$, shall also pass through R.



For join A, R, and produce $\overline{AR}$ to meet BC in D; join, also, E, F; and through R draw (E. 31. 1.) $\overline{PRQ}$ parallel to BC. And, since (*constr.* and E. 2. 6.) $\overline{EF}$ is parallel to $\overline{BC}$, ∴ (E. 29. 1.) the two △ BFE, BRP, are equiangular; as are, also, the △ CEF, CRQ.

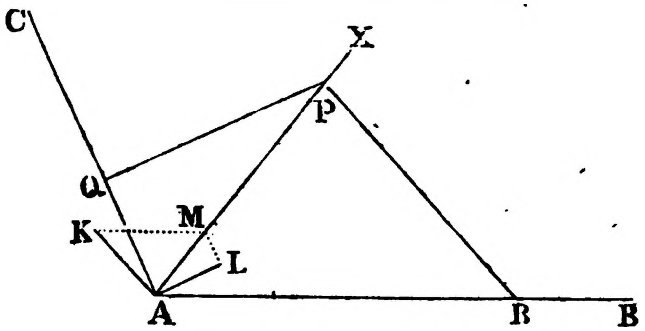
∴ (E. 2. 6.) BF : BR : : CE : CR :
Also (E. 4. 1.) BF : BR : : EF : PR;
And CE : CR : : EF : RQ;
∴ (E. 11. 5.) EF : PR : : EF : RQ;
∴ (E. 9. 5.) PR = RQ;
∴ (S. 61. 1.) BD = DC;

∴, D is the bisection of BC; and there cannot be two straight lines joining the same two points A and D, which do not coincide; ∴ the straight line, drawn from A to the bisection of $\overline{BC}$, passes through the point R.

PROP. VII.

12. PROBLEM. *To find, within a given rectilineal angle, first, the* locus *of all the points, from each of which, if two straight lines be drawn, to the lines containing the given angle, so as always to be parallel to two straight lines given in position, they shall be to one another in a given ratio: And secondly, to find the* locus *of all the points, from each of which if two straight lines be drawn in like manner, they shall cut off from two given parts of the straight lines containing the given angle, segments that shall be to one another in a given ratio.*

Let CAB be the given ∠; let $\overline{AK}$, $\overline{AL}$, be the two



straight lines given in position; and let $\overline{AL}$ be to $\overline{AK}$ in the given ratio: It is required, first, to find, within the ∠ CAB, the *locus* of all the points, from which, if straight lines be drawn to $\overline{AC}$ and $\overline{AB}$, parallel to AL and AK, respectively, they shall be to one another as AL to AK.

Through K and L draw (E. 31. 1.) $\overline{KM}$, and $\overline{LM}$, parallel to $\overline{AB}$ and $\overline{AC}$, respectively, and meeting in M; draw $\overline{AM}$, and produce it, indefinitely, toward X; $\overline{AX}$ is the *locus* which was to be found.

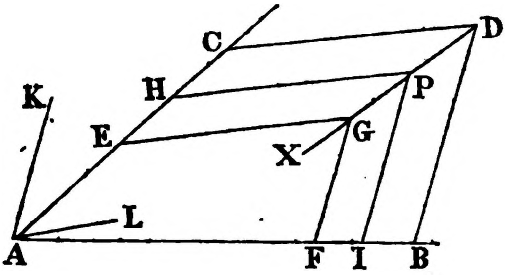
For take any point P in $\overline{AX}$, and from P draw $\overline{PQ}$ parallel to $\overline{AL}$, and $\overline{PR}$ parallel to $\overline{AK}$: And, since (*constr*. and E. 29. 1.) the △ APR, KAM are equiangular, as are, likewise, the △ APQ, MAL.

∴ (E. 4. 6.) PR : AK : : AP : AM : : PQ : AL

∴ (E. 11. 5.) PR : AK : : PQ : AL

∴ (E. 16. 5.) PR : PQ : : AK : AL.

Secondly, let B and C be two given points in AB and AC: It is required to find the *locus* of



all the points, from which if straight lines be drawn parallel to AK and AL, they shall cut off from CA and BA two segments, which are always to one another in the same ratio as the given finite straight lines AK and AL.

From CA cut off CE = AK, and from BA cut off BF = AL; from C and B, draw (E. 31. 1.) CD parallel to AL, and BD parallel to AK, and let CD and BD meet in D; likewise from E and F draw EG parallel to AL or CD, and FG parallel to AK or BD, and let EG and FG meet in G: Through D and G draw the straight line $\overline{\text{DGX}}$: Then is $\overline{\text{DGX}}$ the *locus* which, in this case, was to be found.

For take any point in it, as P, and draw PH parallel to DC, and PI parallel to DB: Then it is manifest from the demonstration of E. 10. 6. that

HC : EC : : PD : GD : : IB : FB;
∴ (E. 16. 5.) HC : IB : : EC : FB :

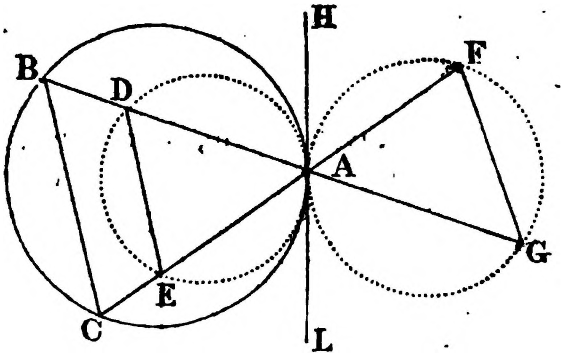
That is (*constr.*) HC is to IB in the given ratio: And it is easily shewn, *ex absurdo*, that no point which is out of the *locus* so determined, has the property described in the proposition.

13. Cor. The intersection of the one *locus* with the other, determines a point, from which if two straight lines be drawn to AB and AC, in the given directions, they shall be to one another in the same given ratio as the segments are, which they cut off from CA and BA.

Prop. VIII.

14. Theorem. *If a circle be touched, in the same point, both externally and internally, by two other circles, and through the point of contact two straight lines be drawn, the parts of them intercepted between the circumference of the given circle, and that of the circle which touches it internally, shall have to one another the same ratio as the parts which are chords of the other circle.*

Let the given circle ABC be touched in the same point A, internally by the circle DAE, and externally by the circle FAG; and through A let there be drawn any two straight lines, BAG, CAF,



each cutting the three circles ABC, DAE, FAG: Then BD:CE::AG:AF.

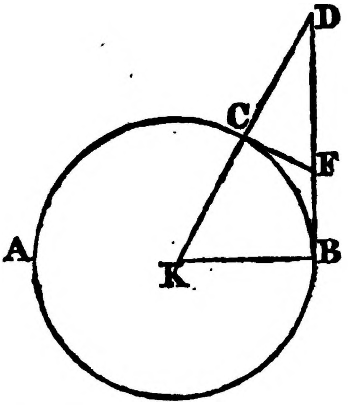
For, draw $\overline{BC}$, $\overline{DE}$, and $\overline{FG}$; and through A draw (E. 17. 1.) HAL touching the circle BAC, in A, and ∴ touching the two circles DAE, FAG: And since (E. 15. 1.) the $\angle DAH = \angle LAG$, and that (E. 32. 3.) the $\angle DAH = \angle DEA$, and the $\angle LAG = \angle AFG$, ∴ the $\angle DEF = \angle EFG$, and ∴ (E. 27. 1.) $\overline{FG}$ is parallel to $\overline{DE}$: Also, since (E. 32. 3.) the $\angle DAH$ or BAH, is equal to each of the $\angle\angle$ DEA, BCA, they are equal to one another, and ∴ (E. 28. 1.) $\overline{BC}$ is parallel to $\overline{DE}$; ∴ (E. 2. 6.) BD:CE::AG:AF.

PROP. IX.

15. PROBLEM. *From the centre of a given circle, to draw a straight line to meet a given tangent to the circle, so that the segment of the line between*

the circle and the tangent shall be any required part of the tangent.

Let ABC be a given circle, of which K is the



centre, and let $\overline{BD}$ touch the circle in B: It is required to draw a straight line from K to $\overline{BD}$, so that the segment of it, between the circle and BD shall be any required part of the segment BD.

Draw $\overline{KB}$; divide (S. 49. 1.) $\overline{KB}$ into a number of equal parts, equal to the number of times which the segment of $\overline{BD}$ is to contain the segment of the straight line to be drawn from K to $\overline{BD}$; and from BD cut off BF equal to one of them; from F draw (E. 17. 3.) $\overline{FC}$ touching the circle ABC in C; through C draw $\overline{KCD}$: Then shall $\overline{CD}$ be the required part of $\overline{BD}$.

For (*constr.* and S. 26. 1.) the two △s KBD, DCF, are equiangular; also (*constr.* and S. 19. 3. *cor.* 1.) $\overline{FB}=\overline{FC}$;

∴ (E. 4. 1.) KB:BD::CF or BF:CD;

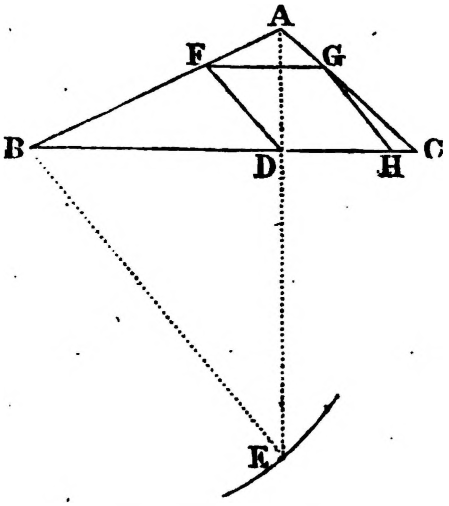
∴ (E. 16. 5.) KB:BF ::BD:CD;

∴ (*constr.* and S. 4. 5.) CD is the required part of BD.

PROP. X.

16. PROBLEM. *From a given triangle to cut off a rhombus; the base of the rhombus being part of the base of the triangle, and having its extremity in a given point of that base.*

Let ABC be the given △, and D the given point in its base BC: It is required to cut off from the △ABC a rhombus, having its base in $\overline{BC}$, and terminated by the given point D.



Draw $\overline{AD}$, and produce it; from the centre B, at the distance BC, describe a circle, cutting AD produced in E, and join B, E; ∴ $\overline{BE} = \overline{BC}$; through D draw (E. 31. 1.) $\overline{DF}$ parallel to $\overline{EB}$; also through F draw FG parallel to BC, and through G draw GH parallel to $\overline{FD}$ or $\overline{BE}$; ∴ the figure FDHG is a ▭: And since (*constr.* and E. 28. 1.) the △ BAC, FAG, are equiangular, as are, also, the △ ABE, AFD,

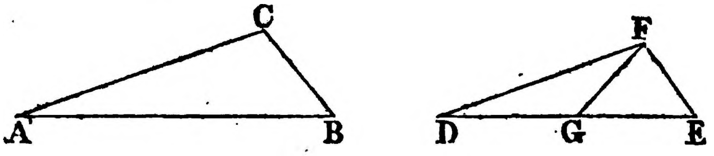
∴ (E. 4. 6.) $AB : BC$ or $BE :: AF : FG$:
And $AB : BE :: AF : FD$;
∴ (E. 11. 5.) $AF : FG :: AF : FD$;
∴ (E. 9. 5.) $FG = FD$:

But (E. 34. 1.) $FG = DH$, and $FD = GH$; ∴ the figure FDHG, having its base DH in BC, and terminated by the given point D, is a rhombus.

PROP. XI.

17. THEOREM. *If two triangles have one angle of the one, equal to one angle of the other, and also another angle of the one, together with another angle of the other, equal to two right angles, the sides about the two remaining angles shall be proportionals.*

Let the two △ ABC, DEF, have the ∠ BAC = ∠ EDF, and another ∠, as ACB, of the one △, together with another ∠, as DEF, of the other,



equal to two right angles: Then AB : BC :: DF : FE.

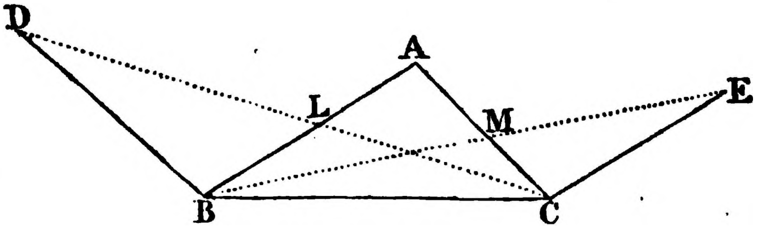
From F draw (S. 25. 1.) $\overline{FG}$ making with $\overline{DE}$ an ∠FGE=∠FEG; ∴ (E. 6. 1.) $\overline{FG}=\overline{FE}$: And since (*hyp.*) the ∠ACB + ∠DEF = two right ∠∠, and that (E. 13. 1.) the ∠DGF+∠FGE= two right ∠∠, ∴ the ∠ACB + ∠DEF= ∠DGF + ∠FGE; but (*constr.*) the ∠FGE= ∠FEG; ∴ the ∠ACB= ∠DGF; and (*hyp.*) the ∠BAC=∠GDF; ∴ (S. 26. 1.) the two △ ACB, DGF, are equiangular; ∴ (E. 4. 1.) AB : BC :: DF : FG or FE.

PROP. XII.

18. THEOREM. *If, from the extremities of the base of a given triangle, there be drawn two straight lines, both on the same side of the base, and each equal to the adjacent side, and making with that side an angle equal to the vertical angle of the triangle, then the straight lines which join the extremities of the lines so drawn, and the further*

extremities of the base, shall cut off, from the sides, equal segments towards the vertex; and each of those segments shall be a mean proportional between the other segments, that are towards the base.

From the extremities B and C of the base BC,



of the $\triangle$ ABC, let $\overline{BD}$ be drawn (E. 31. 1.) parallel to $\overline{AC}$, and made equal to $\overline{AB}$; and let $\overline{CE}$ be drawn parallel to $\overline{AB}$, and made equal to $\overline{AC}$; so that (E. 29. 1.) each of the $\angle$ ABD, ACE, is equal to the vertical $\angle$ BAC; also, let $\overline{DC}$ and $\overline{EB}$ be drawn, cutting $\overline{AB}$ and $\overline{AC}$ in L and M, respectively: Then $\overline{AL} = \overline{AM}$; and BL : LA :: AM or LA : MC.

For (*constr.* and E. 15. 1.) the $\triangle$ DLB, ALC, are equiangular, as are, also, the $\triangle$ EMC, AMB;

∴ (E. 4. 6.) DB or AB : AC :: BL : LA;

∴ (E. 18. 5.) AB + AC : AC :: AB : AL:

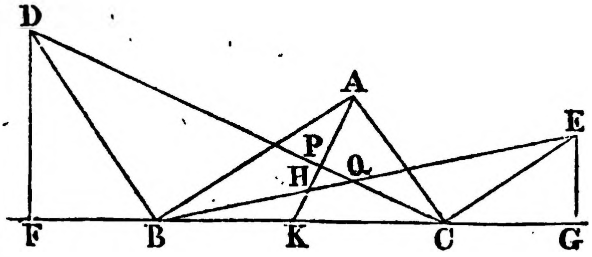
Again (E. 4. 6.) CE or $AC : AB :: CM : MA$;
∴ (E. 18. 5.) $AC + AB : AB :: AC : AM$;
∴ (E. 16. 5.) $AC + AB : AC :: AB : AM$;
∴ (E. 9. 5.) $\overline{AL} = \overline{AM}$.

Also, since it has been shewn, that
$AB : AC :: BL : LA$,
and $AC : AB :: CM : MA$,
∴ (S. 2. 5.) $AB : AC :: AM : CM$;
∴ (E. 11. 5.) $BL : LA :: AM$ or $LA : MC$.

PROP. XIII.

19. **THEOREM.** *If at the extremities of the hypotenuse of a right-angled triangle two straight lines be drawn, on the same side of the hypotenuse as the right angle, each equal to, and each perpendicular to, the adjacent side, the two straight lines joining each of their extremities and the further extremity of the hypotenuse, shall cut each other in the same point of the perpendicular drawn to the hypotenuse from the right angle.*

From the extremities A and B, of the hypotenuse BC of the right-angled △ ABC, let $\overline{BD}$



and $\overline{CE}$ be drawn ⊥ to AB and AC, and equal to AB and AC, respectively, and let $\overline{AK}$ be drawn ⊥ to BC: Then if D, C and E, B be joined, $\overline{DC}$ and $\overline{EB}$ shall cut one another in the same point of $\overline{AK}$.

For, if it be possible, let $\overline{DC}$ cut $\overline{AK}$ in P, and let $\overline{EB}$ cut $\overline{AK}$ in H; and from D and E draw (E. 12. 1.) $\overline{DF}$ and $\overline{EG}$ ⊥ to $\overline{BC}$ produced both ways; ∴ (S. 38. 1.) $\overline{FB}=\overline{GC}$, and ∴ $\overline{FC}=\overline{BG}$: And, since (*constr.*) the ∠∠ PKC, DFC, are right ∠∠, and that the ∠ PCK is common to the two △ PCK, DCF, ∴ (S. 26. 1.) the two △ PCK, DCF are equiangular; and, in the same manner, the two △ HKB, EGB may be shewn to be equiangular; ∴ (E. 4. 6.) CF : FD :: CK : KP.

But (S. 38. 1. *cor.*) $\overline{FD}=\overline{BK}$, and $\overline{CK}=\overline{GE}$; and it has been shewn that $\overline{CF}=\overline{BG}$;

∴ BG : BK :: GE : KP:

But (E. 4. 6.) BG : BK :: GE : KH;

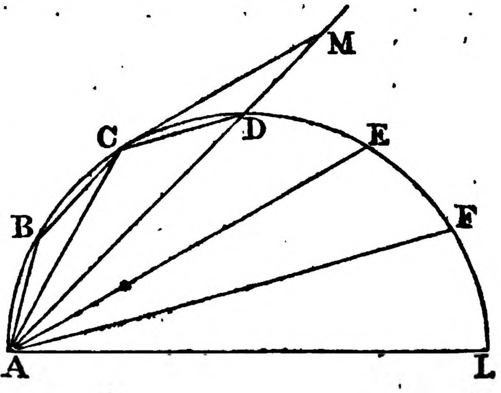
∴ (E. 9. 5.) $\overline{KH}=\overline{KP}$; which is absurd; ∴ $\overline{DC}$ and $\overline{EB}$ cannot cut the perpendicular drawn from A to $\overline{BC}$, in two different points.

PROP. XIV.

20. THEOREM. *The semi-circumference of a circle having been divided into any number of equal parts, and chords having been drawn, from either*

extremity of the diameter, to the several points of division; the first chord has to the second, the same ratio which the second has to the aggregate of the first and third; or the same ratio which any other chord has to the aggregate of the two chords that are next to it.

Let the semi-circumference AEL of a circle, be divided into any number of equal parts, in the points B, C, D, E, F, &c.; and let $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{AE}$, $\overline{AF}$, &c., be drawn: Then



AB : AC :: AC : AB + AD :: AD : AC + AE, and so on.

For, from C, as a centre, at the distance CA, describe a circle cutting $\overline{AD}$, produced, in M, and join B, C, and C, D, and C, M; and since (*hyp.*) $\overparen{AB} = \overparen{BC}$, ∴ (E. 29. 3.) $\overline{AB} = \overline{BC}$; also (E. 27. 3.) the ∠BAC = ∠CAD; ∴ (E. 5. 1. and S. 26. 1.) the isosceles △ABC, ACM, are equiangular;

$\therefore$ (E. 4. 6.) AB : AC :: AC : AM, or DM + AD:
But (E. 22. 3.) since ABCD is a quadrilateral figure inscribed in a circle, the $\angle$ ABC + $\angle$ ADC = two right $\angle\angle$; also (E. 13. 1.) the $\angle$ ADC + $\angle$ CDM = two right $\angle\angle$; $\therefore$ the $\angle$ CDM = $\angle$ ABC; and the $\angle$ BAC = $\angle$ CAD, or (*constr.* and E. 5. 1.) $\angle$ CMD; and the side CM, of the $\triangle$CDM, is equal to the side CA, of the $\triangle$ ABC; $\therefore$ (E. 26. 1.) $\overline{\text{DM}} = \overline{\text{AB}}$; and it has been shewn that

AB : AC :: AC : DM + AD;

$\therefore$ AB : AC :: AC : AB + AD:

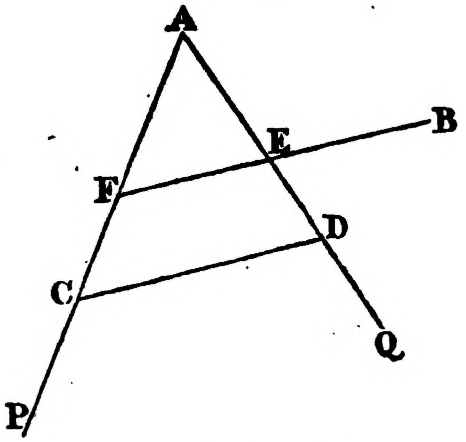
And, by a similar construction, and a similar method of proof, the remaining part of the proposition may be demonstrated.

Prop. XV.

21. Problem. *From a given point, either within or without a given rectilineal angle, to draw a straight line cutting off from the lines which contain the angle, segments, towards the summit of the angle, which shall be to one another in a given ratio.*

Let PAQ be the given $\angle$, and first let B be a given point without it: It is required to draw, from B, a straight line which shall cut off from $\overline{\text{AP}}$ and $\overline{\text{AQ}}$, two segments, towards A, which shall be to one another in a given ratio.

From $\overline{\text{AP}}$ and $\overline{\text{AQ}}$ cut off AC and AD, equal



to the two straight lines which exhibit the given ratio, each to each; join D, C; and through B draw (E. 31. 1.) $\overline{\text{BEF}}$ parallel to $\overline{\text{DC}}$: Then, since (*constr.* and E. 29. 1.) the two △ ADC, AEF are equiangular,

∴ (E. 4. 6.) AF : AE :: AC : AD;

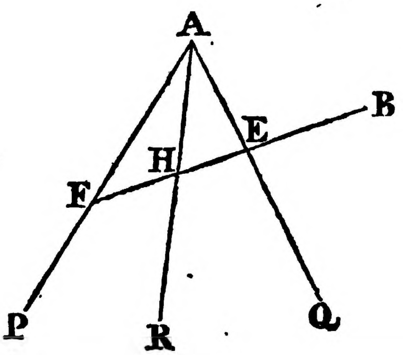
∴ (*constr.*) AF is to AE in the given ratio.

And the problem may be solved in the same manner, when the given point is within the given angle.

PROP. XVI.

22. PROBLEM. *To draw through a given point a straight line cutting the lines which contain a given rectilineal angle, so that the segment of it, between those lines, shall be divided by the straight line that bisects the given angle, into two parts, which are to one another in a given ratio.*

Let PAQ be the given ∠; let $\overline{AR}$ be drawn (E. 9. 1.) bisecting it; and let B be the given point: It is required to draw, from B, a straight line cutting $\overline{AP}$ and $\overline{AQ}$, so that the segment of it, between $\overline{AP}$ and $\overline{AQ}$, shall be divided by $\overline{AR}$, into two parts, which are to one another in a given ratio.

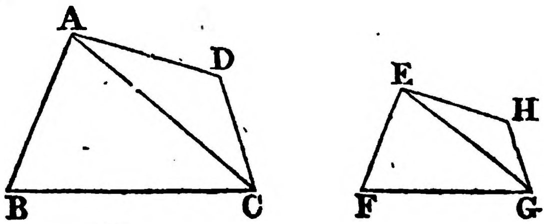


Draw (S. 15. 6.) $\overline{BEF}$, so that AF shall be to AE in the given ratio, and let BF cut AR in H; then, (E. 3. 6.) since $\overline{AH}$ bisects the ∠ FAE, FH : HE :: AF : AE; that is, FG is to GE in the given ratio.

PROP. XVII.

23. THEOREM. *If two trapeziums have an angle of the one equal to an angle of the other, and if, also, the sides of the two figures, about each of their angles, be proportionals, the remaining angles of the one shall be equal to the remaining angles of the other.*

Let the two trapeziums ABCD, EFGH, which



have the sides about each of their ∠s proportionals, have the ∠ ABC equal to the ∠ EFG: The two figures shall be equiangular.

For draw $\overline{AC}$ and $\overline{EG}$: Then (*hyp.* and E. 6. 6.) the △s ABC, EFG, are equiangular, and have their equal angles opposite to the homologous sides.

∴ (E. 4. 6.) BA : AC :: FE : EG;
and (*hyp.*) DA : BA :: HE : FE;
∴ (E. 22. 5.) DA : AC :: HE : EG:

And in the same manner it may be shewn, that
DC : CA :: HG : GE:
And (*hyp.*) AD : DC :: EH : HG;
∴ (E. 5. 6.) the △s ADC, EHG, are equiangular, and have their equal ∠s opposite to the homologous sides; and it has been shewn, that the △s ABC, EFG, are likewise equiangular; ∴ the trapeziums ABCD, EFGH are equiangular.

PROP. XVIII.

24. THEOREM. *If two straight lines touch a circle at opposite extremities of its diameter, any other*

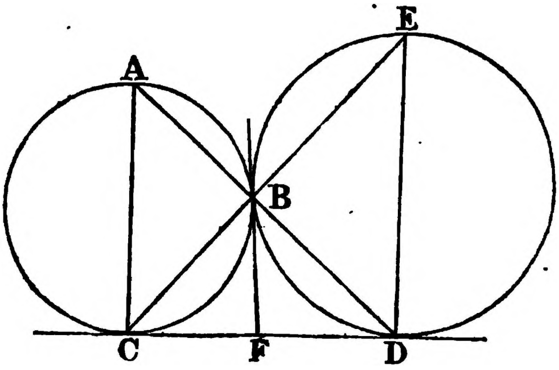
tangent of the circle, terminated by them, is so divided in its point of contact, that the radius of the circle is a mean proportional between its segments.

For (S. 20. 3. *cor.*) the tangent, so terminated, subtends at the centre of the circle a right ∠, and (E. 18. 3.) the straight line drawn from the centre to the point of contact, meets that tangent at right ∠∠, and is, ∴, (E. 8. 6. *cor.*) a mean proportional between the segments of the tangent.

Prop. XIX.

25. **Theorem.** *If two given circles touch each other, and also touch a given straight line, the part of the line between the points of contact, is a mean proportional between the diameters of the circles.*

Let the two circles ABC, EBD, which touch



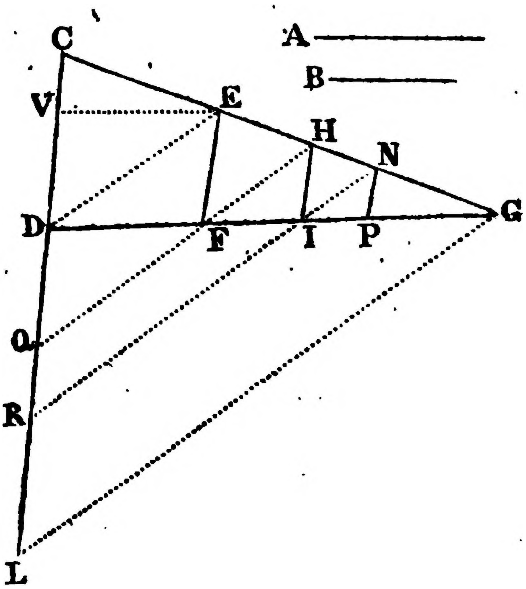
one another in B, be each of them touched by $\overline{CD}$ in the points C and D: Then is $\overline{CD}$ a mean proportional between the diameters of the two circles ABC, EBP.

For draw the diameters CA and DE, which (E. 18. 3.) are ⊥ to $\overline{CD}$; also draw (E. 17. 3.) BF touching each of the circles in B, and join A, B, and C, B, and E, B, and D, B: Then, since (S. 19. 3. *cor.* 2.) $\overline{FB} = \overline{FC}$, and $\overline{FB} = \overline{FD}$, a circle described from the centre F, at the distance FB, would pass through C and D; ∴ (E. 31. 3.) the ∠CBD is a right ∠, as is, also, the ∠EBD; ∴ (E. 14. 1.) $\overline{CB}$ and $\overline{BE}$ are in the same straight line; and, in the same manner, it may be shewn that $\overline{AB}$ and $\overline{BD}$ are in the same straight line; but (E. 8. 6.) the ∠CAD = ∠DCB or DCE; ∴ (S. 26. 1.) the two right-angled △ EDC, DCA, are equiangular; ∴ (E. 4. 6.) ED:DC::DC:CA.

PROP. XX.

26. **PROBLEM.** *Two straight lines being given, which are the two first of a series of proportionals, to find the rest; and, if the series decrease, to find a line which shall be greater than the aggregate of any number, whatever, of its terms, but to which the aggregate may approximate indefinitely.*

Let A, B be the two first of a decreasing series of proportionals: It is required to find a line



which shall be the limit of the aggregate of the proportionals.

Make $\overline{CD}=A$, and $\overline{EF}=B$; and let $\overline{EF}$ be drawn (E. 31. 1.) parallel to $\overline{CD}$; join C, E and D, F, and let $\overline{CE}$ and $\overline{DF}$ be produced, so as to meet, in G; join E, D, and through G draw $\overline{GL}$ parallel to $\overline{ED}$, and let it meet $\overline{CD}$, produced, in L: Then is $\overline{CL}$ the line which was to be found.

For, through F draw $\overline{HFQ}$ parallel to $\overline{ED}$; and through H draw HI parallel to $\overline{EF}$ or $\overline{CD}$:

Then (*constr.* and E. 34. 1.) $\overline{DQ}=\overline{EF}$; also, (*constr.* and E. 2. 6.)

$$CD:DQ::CE:EH::DF:FI:$$

But since (*constr.* and E. 27. 28. 1.) the △EFD, HIF are equiangular.

∴ (E. 4. 6.) DF : FI :: EF : HI;
∴ (E. 11. 5.) CD : DQ or EF :: EF : HI:

So that $\overline{HI}$ is the next of the proportionals to EF; and, by a similar construction, the next of them $\overline{NP}$, may be found; and so on: But $\overline{CL} = \overline{CD} + \overline{DQ} + \overline{QR}$, + &c.; and, by the construction, DQ, QR, &c. are equal to the several proportionals: It is manifest, ∴, that $\overline{CL}$ is their limit.

27. Cor. The first term of a decreasing series of proportionals is a mean between the excess of the first term above the second, and the line which is the limit of all the terms.

For draw $\overline{EV}$ parallel to $\overline{DG}$; then since (E. 29. 1.) the △ CVE, CDG are equiangular, as are, also, the △ CED, CGL,

∴ (E. 4. 6.) CV : CD :: CE : CG :: CD : CL;
∴ (E. 11. 1.) CV : CD :: CD : CL.

And, since (E. 34. 1.) VD = EF, ∴ CV = CD − EF.

Prop. XXI.

28. Problem. *To describe a square which shall have a given ratio to a given rectilineal figure.*

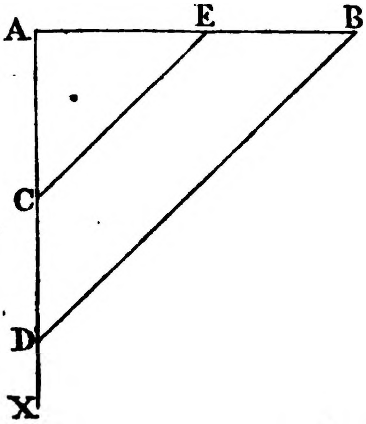
Find (E. 14. 2.) a square that shall be equal to the given rectilineal figure, and from its side, produced if it be necessary, cut off (E. 10. 6.) a part, which shall be to the side itself in the given ratio: The rectangle, contained by the side of the square and the part so cut off, will (E. 1. 6.) have to the given square the given ratio: If, ∴, lastly, a square be found (E. 14. 2.) that is equal to the rectangle, it will have to the given square the given ratio.

29. Cor. Hence a square may be cut off from a given square, which shall be any required part of it.

Prop. XXII.

30. Problem. *To divide a given finite straight line into two parts, the squares of which shall be to one another in a given ratio.*

Let AB be the given finite straight line: It is



required to divide it into two parts, the squares of which shall be to one another in a given ratio.

Draw (E. 11. 1.) $\overline{AX} \perp$ to $\overline{AB}$; find (S. 21. 6.) the sides of two squares, which shall be to one another in the given ratio, and from AX cut off $\overline{AC}$ equal to one of them, and $\overline{CD}$ equal to the other; join D, B; and from C draw (E. 31. 1.) $\overline{CE}$ parallel to $\overline{DB}$: Then is $\overline{AB}$ divided in E, so that the squares of AE and EB are to one another in the given ratio.

For (*constr.* and E. 2. 6.) AE : EB :: AC : CD;

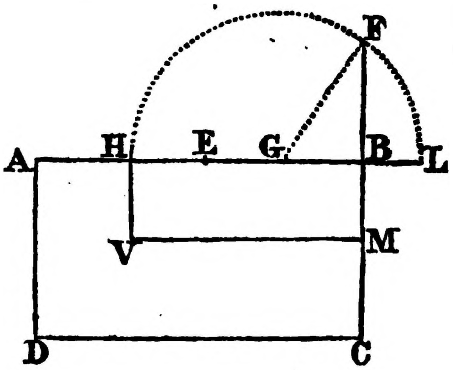
∴ (S. 1. 6. *cor.* 1.) $\overline{AE}^2 : \overline{EB}^2 :: \overline{AC}^2 : \overline{CD}^2$:

But (*constr.*) $\overline{AC}^2$ is to $\overline{CD}^2$ in the given ratio; ∴ $\overline{AE}^2$ is to $\overline{EB}^2$ in the given ratio.

PROP. XXIII.

31. **PROBLEM.** *To find two points, situated in two adjacent sides of a given oblong, at equal distances from two opposite angles, from which, if two straight lines be drawn parallel to the sides of the figure, they shall cut off from it any part required.*

Let ABCD be a given oblong: It is required to



find in two of its adjacent sides, as in AB and BC, two points equidistant from the ∠s A and C, from which if straight lines be drawn parallel to $\overline{BC}$ and $\overline{BA}$, they shall cut off a given part of the oblong ABCD.

From $\overline{AB}$ cut off $\overline{AE}=\overline{BC}$; produce $\overline{CB}$; find (S. 21. 6.) a square which shall be the same part of the given oblong, as that which is to be cut off, and in $\overline{CB}$, produced, make $\overline{BF}$ equal to the side of that square; bisect $\overline{EB}$ in G; from the centre G, at the distance $\overline{GF}$, describe the circle HFL, cutting $\overline{AB}$ produced in L, and $\overline{AB}$ in H; from $\overline{CB}$ cut off $\overline{CM}=\overline{AH}$: Then are H and M the points which were to be found.

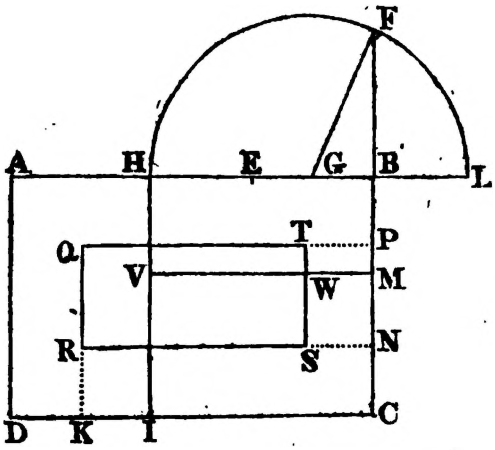
For, since (*constr.*) $\overline{AE}=\overline{BC}$, and $\overline{AH}=\overline{CM}$, ∴, $\overline{HE}=\overline{BM}$: Again, since (*constr.*) $\overline{HG}=\overline{GL}$ and $\overline{EG}=\overline{GB}$, ∴ $\overline{HE}=\overline{BL}$; and it has been shewn that $\overline{HE}=\overline{BM}$; ∴ $\overline{BL}=\overline{BM}$; but (*constr.* and E. 35. 3.) $\overline{HB}\times\overline{BL}=\overline{BF}^2$; ∴ $\overline{HB}\times\overline{BM}=\overline{BF}^2$; ∴ (*constr.*) $\overline{HB}\times\overline{BM}$ is the required part of $\overline{AB}\times\overline{BC}$; and the points H and M are equidistant from A and C.

PROP. XXIV.

32. PROBLEM. *Within a given oblong, to describe another oblong which shall be any required part of*

it, and shall have its four sides all equally distant from the four sides of the given rectangle.

Let ABCD be a given oblong: It is required



to describe within it another oblong, which shall be a given part of ABCD, and have its sides equally distant from the sides of ABCD, each from each.

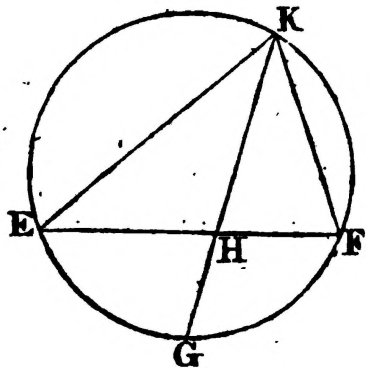
From ABCD cut off (S. 23. 6.) the oblong HBMV, the same part of it as that which is to be described, is required to be, and having the extremities H and M, of its sides BH and BM, equally distant from A and C; let $\overline{HV}$, produced, meet $\overline{DC}$ in I; bisect (E. 10. 1.) $\overline{DI}$ in K, and $\overline{CM}$ in N; ∴ $\overline{DK} = \overline{CN}$; from K draw (E. 31. 1.) $\overline{KQ}$ parallel to $\overline{BC}$, and from N draw $\overline{NR}$ parallel to $\overline{AB}$; from $\overline{BC}$ and $\overline{NR}$ cut off $\overline{BP}$ and $\overline{NS}$, each equal to $\overline{DK}$ or $\overline{CN}$; through P draw

$\overline{PQ}$ parallel to $\overline{AB}$, and through S draw $\overline{ST}$ parallel to BC; ∴ the figure QRST is an oblong: And it is manifest, from the construction, that $\overline{RS}=\overline{HB}$ and $\overline{RQ}=\overline{BM}$, and that, ∴, the gnomon QRW is equal to the gnomon HBW, for the one may evidently be applied to the other so as to coincide with it; add to these equals the rectangle VT, and it is plain that the oblong QRST is equal to the oblong HM, which was made the required part of the given oblong ABCD.

Prop. XXV.

33. **Problem.** *The base, the vertical angle, and the ratio of the two sides of a triangle being given, to construct it.*

Let EF be a given straight line: Upon EF, as



a base, it is required to construct a △, having its vertical $\angle$ equal to a given $\angle$, and its two remaining sides in a given ratio to one another.

Upon $\overline{EF}$ describe (E. 33. 3.) a segment of a circle EKF, capable of containing an $\angle$ equal to the given $\angle$, and complete the circle EKFG; divide (E. 10. 6.) $\overline{EF}$ in H, so that $\overline{EH}$ is to $\overline{HF}$ in the given ratio; bisect (E. 30. 3.) $\overset{\frown}{EGF}$ in G; draw $\overline{GH}$, and produce it to meet the circumference in K; lastly, join E, K, and F, K: Then is EKF the $\triangle$ which was to be constructed.

For since (*constr.*) $\overset{\frown}{EG} = \overset{\frown}{FG}$, $\therefore$ (E. 27. 3.) the $\angle$ EKG $=$ $\angle$ FKG, so that the $\angle$ EKF is bisected by $\overline{KH}$;

$\therefore$ (E. 3. 6.) KE : KF :: EH : HF:

That is (*constr.*) KE is to KF in the given ratio, and the vertical $\angle$ EKF is equal to the given angle.

PROP. XXVI.

34. PROBLEM. *A given finite straight line being divided into any two given parts, to divide it again, so that the rectangle contained by the two former given parts shall have a given ratio to the rectangle contained by the two latter parts.*

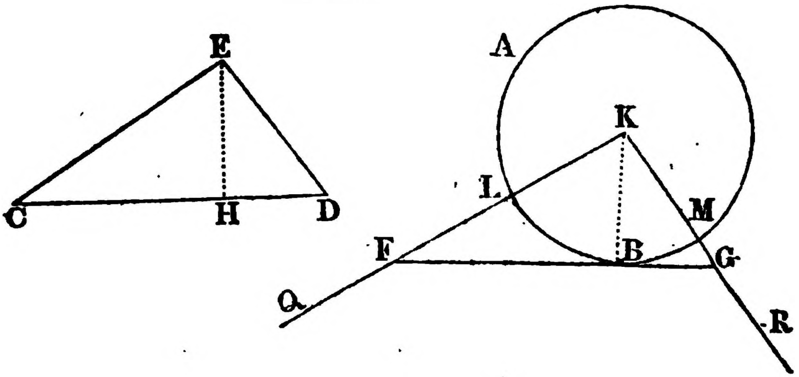
Describe (S. 21. 6.) a square which shall be to the rectangle, contained by the given parts of the given line, in the given ratio; and divide (S. 71. 3.)

the given line into two parts, so that the rectangle contained by them shall be equal to the square so described: It is manifest that this rectangle will be to the rectangle, contained by the two given parts, in the given ratio.

Prop. XXVII.

35. Problem. *To draw a straight line to touch a given arch of a circle, so that being terminated by the semi-diameters, produced, which bound the arch, it shall be divided by the point of contact, into two parts that are to one another in a given ratio.*

Let LBM be a given arch of the circle ALBM,



terminated by the two semi-diameters KL and KM: It is required to draw a tangent to the circle, so that, being terminated by KL and KM, produced, it shall be divided, by the point of its contact, into two segments, that are to one another in a given ratio.

Take any straight line CD, and divide it (E. 10. 6.) in H in the given ratio; draw (E. 11. 1.) $\overline{HE}$ ⊥ to $\overline{CD}$, and let $\overline{HE}$ be cut in E, by a segment of a circle described (E. 33. 3.) upon $\overline{CD}$, capable of containing an ∠ equal to the given ∠ LKM; and join C, E, and D, E; ∴ the ∠ CED = ∠ LKM; lastly, draw (S. 8. 3. *cor.*) $\overline{FBG}$ touching the circle ALBM, and making with $\overline{KL}$, produced, an ∠ KFG = ∠ ECD: Then is the tangent FG divided in B, so that FB is to BG in the given ratio.

For join K, B; ∴ (*constr.* and E. 18. 3.) the ∠∠ at B are right ∠∠; as are, also, the ∠∠ at H; and (*constr.*) the ∠ ECH = ∠ KFB; ∴ (S. 26. 1.) the △△ ECH, KFB are equiangular; and since the ∠ CEH = ∠ FKB, and that (*constr.*) the whole ∠ CED = whole ∠ FKG, ∴ the ∠ HED = ∠ BKG, and (S. 26. 1.) the △△ EHD, KBG are equiangular, as are, likewise, the △△ CED, FKG;

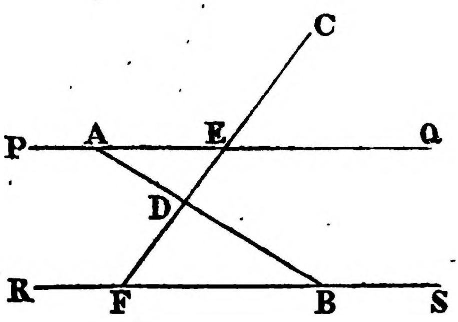
∴ (E. 4. 6.) CH : HE : : FB : BK;
and HE : HD : : BK : BG;
∴ (E. 22. 5.) CH : HD : : FB : BG:

But (*constr.*) CH is to HD in the given ratio; ∴ FB is to BG in the given ratio.

PROP. XXVIII.

36. **PROBLEM.** *Two points being given, one in each of two parallel straight lines, and a third point being also given, without them, to draw, from that third point, a straight line so to cut the parallels, as that the segments of the parallels, between it and the two first points, shall be to one another in a given ratio.*

Let PQ and RS be the two given parallel straight lines; A and B the two given points in them; and C a given point without them: It is required to draw from C a straight line cutting PQ and RS, so that the segments of PQ and RS, between the cutting line and the given points A and B, shall be to one another in a given ratio.



Join A, B; and divide (E. 10. 6.) AB in D, so that AD is to DB in the given ratio; through D draw CEF, cutting PQ and RS, in E and F: Then is $\overline{\text{CEF}}$ the straight line which was to be drawn.

For, since $\overline{\text{PQ}}$ is (*hyp.*) parallel to $\overline{\text{RS}}$, $\therefore$ (E. 29. 1.) the $\angle$ AEF $=\angle$ EFB; and the $\angle$ EAB $=\angle$ ABF; also (E. 15. 1.) the $\angle$ ADE $=\angle$ FDB; so that the $\triangle$s ADE, BDF are equiangular;

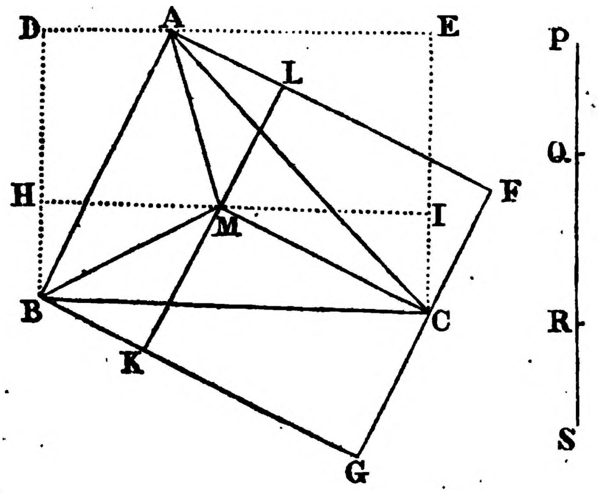
$\therefore$ (E. 4. 6.) AE : BF :: AD : DB :

But (*constr.*) AD is to DB in the given ratio; $\therefore$ AE is to BF in the given ratio.

PROP. XXIX.

37. PROBLEM. *To find a point within a given triangle, from which if three straight lines be drawn to the three angles of the triangle, it shall thereby be divided into three parts that are each to each in given ratios.*

Let ABC be the given $\triangle$, and let $\overline{\text{PQ}}$, $\overline{\text{QR}}$, $\overline{\text{RS}}$,



placed in the same straight line, be three given straight lines: It is required to find a point within the △ ABC, from which if straight lines be drawn to A, B and C, the △ shall thereby be divided into three parts that are to one another as $\overline{PQ}$, $\overline{QR}$, and $\overline{RS}$.

Through A draw (E. 31. 1.) $\overline{DAE}$ parallel to $\overline{BC}$, and from B and C draw (E. 11. 1.) $\overline{BD}$ and $\overline{CE}$ ⊥ to $\overline{BC}$; in like manner, describe upon $\overline{AB}$ another rectangle ABGF, about the △ ABC; divide (E. 10. 6.) $\overline{DB}$ in H, so that PS : PQ : : DB : BH; divide, also, $\overline{BG}$ in K, so that PS : QR : : BG : BK; through H draw $\overline{HI}$ parallel to $\overline{BC}$, and through K draw $\overline{KL}$ parallel to $\overline{BA}$, and let $\overline{HI}$ and $\overline{KL}$ cut one another in M: Then is M the point which was to be found.

For draw $\overline{MA}$, $\overline{MB}$, and $\overline{MC}$: And since (E. 41. 1.) each of the rectangles DBCE, ABGF, is double of the △ ABC, they are equal to one another; also (E. 41. 1.) AK is double of the △ AMB, and HC is double of the △ BMC: But (E. 1. 6.)

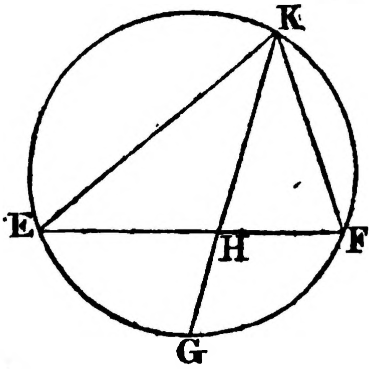
HBCI : DBCE : : HB : DB : : PQ : PS;
and ABGF : ABKL : : KB : GB : : PS : QR;
∴ (E. 22. 5.) HBCI : ABKL : : PQ : QR;
∴ (E. 41. 1. and E. 15. 5.)
△ BMC : △ AMB : : PQ : QR:

Whence it follows, also, that the △ AMB : △ AMC : : QR : RS.

PROP. XXX.

38. **PROBLEM.** *To divide a given circular arch into two parts, so that the chords of these parts shall be to each other in a given ratio.*

Let EKF be the given circular arch: It is required to divide it into two parts, the chords of which shall be to one another in a given ratio.



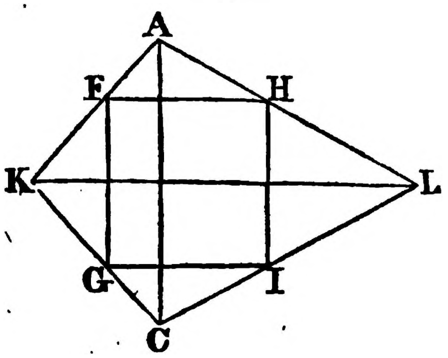
Join E, F; and describe (E. 25. 3.) the circle KEGF, of which EKF is a given segment; bisect (E. 30. 3.) $\overset{\frown}{EGF}$ in G; divide (E. 10. 6.) $\overline{EF}$ in H, so that $\overline{EH}$ shall be to $\overline{HF}$ in the given ratio; draw $\overline{GH}$, and produce it to meet the circumference in K; lastly join E, K and F, K.

Then, since (*constr.* and E. 27. 3.) the ∠ EKF is bisected by $\overline{KHG}$, ∴ (E. 3. 6.) $\overline{KE} : \overline{KF} : : \overline{EH} : \overline{HF}$; that is, (*constr.*) $\overline{KE} : \overline{KH}$ in the given ratio.

PROP. XXXI.

39. PROBLEM. *To inscribe a square in a given trapezium, which has the two sides about any angle equal to one another, and the two sides about the opposite angle also equal to one another.*

Let AKCL be a trapezium having the side



KA = KC, and also the side LA = LC: It is required to inscribe in AKCL a square.

Draw the diameters of the figure, AC and KL; divide (E. 10. 6.) $\overline{AK}$ in F, so that AF : FK :: AC : KL; draw (E. 31. 1.) $\overline{FG}$ parallel to $\overline{AC}$, and $\overline{GI}$ and $\overline{FH}$ parallel to $\overline{KL}$; and join H, I: Then is the inscribed figure FHIG a square.

For (S. 1. 3. *cor.*) $\overline{KL}$ bisects $\overline{AC}$ at right ∠∠; ∴ (*constr.* and E. 34. 1.) the ∠∠ at F and G are right ∠∠: Again the △△ AFH, AKL (E. 29. 1.)

are equiangular, as are, also, the △ KFG, KAC;

∴ (E. 4. 6.) AK : KL : : AF : FH :
And (*constr.*) KL : AC : : KF : AF;
∴ (E. 23. 5.) AK : AC : : KF : FH :
But (E. 4. 6.) AK : AC : : KF : FG;
∴ (E. 9. 5.) FG = FH :

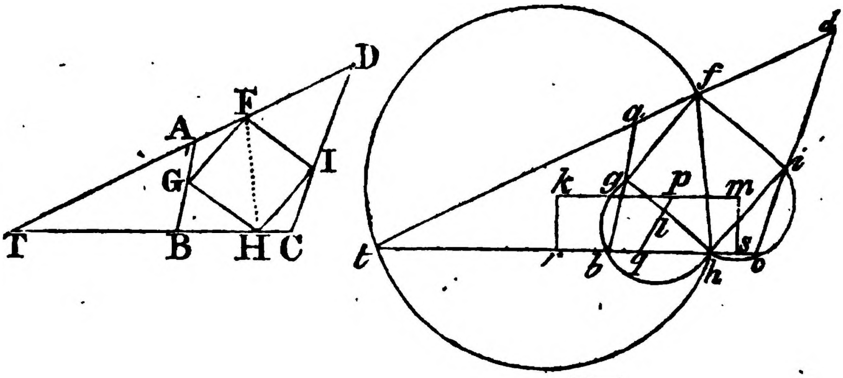
And since (E. 2. 6.) CG : GK : : AF : FK, it may, in like manner, be shewn that $\overline{GI} = \overline{GF}$; and (*constr.*) $\overline{GI}$ is parallel to $\overline{FH}$; ∴ (E. 33. 1.) IH is equal and parallel to GF; ∴ the figure FHIG is an equilateral ▭; and its ∠ GFH, FGI, have been shewn to be right ∠; ∴ (E. 34. 1.) all its ∠ are right ∠; ∴ (E. 30. def. 1.) FHIG is a square.

PROP. XXXII.

40. PROBLEM. *To inscribe a square in a given trapezium.*

Let ABCD be the given trapezium: It is required to inscribe in it a square.

Since (E. 34. def. 1.) ABCD is not a ▭, one pair, at least, of its opposite sides must meet if they be far enough produced; let, ∴, DA and CB be produced so as to meet in T: Take any straight line *fg* and upon it describe (E. 46. 1.) the square *fghi*; join *f*, *h*; and upon $\overline{hf}$, $\overline{hg}$, and $\overline{hi}$ describe (E. 33. 3.) segments of circles, *ich*, *fth*, and *gbh*, capable of containing ∠ equal, respectively, to the ∠ T, B, and C, and let *k*, *l*, and *m*, be the se-



veral centres of the circles; draw $\overline{km}$, and divide it (E. 10. 6.) in p, so that $mp : pk :: \text{CB} : \text{BT}$; also join p, l; through h draw (E. 12. 1.) $\overline{chq} \perp$ to $\overline{pl}$ produced, and meeting it in q; also let $\overline{cq}$, produced, meet the circumference fth in t, the circumference gbh in b, and the circumference ich in c: Again, divide (E. 10. 6.) $\overline{\text{BC}}$ in H, so that $\text{BH} : \text{HC} :: bh : hc$; make (E. 23. 1.), at the point H, in $\overline{\text{BH}}$, the $\angle\, \text{BHG} = \angle\, bhg$, the $\angle\, \text{BHF} = \angle\, bhf$, and the $\angle\, \text{CHI} = \angle\, chi$; lastly, join F, G and F, I: Then is the inscribed figure FGHI a square.

For draw (E. 12. 1.) $\overline{kr}$ and $\overline{pq}$, $\perp$ to $\overline{tc}$: Then, since (*constr.* and E. 3. 3.) $\overline{bh} = 2\overline{qh}$, and $\overline{hc} = 2\overline{hs}$, it is manifest that $\overline{bc} = 2\overline{qs}$; and, in the same manner, it may be shewn that $\overline{tb} = 2\overline{rq}$;

$$\therefore \text{(E. 15. 5.)}\ tb : bc :: rq : qs:$$

But (*constr.* and E. 10. 6.)

$$rq : qs :: kp : pm :: \text{TB} : \text{BC};$$

$$\therefore \text{(E. 11. 5.)}\ tb : bc :: \text{TB} : \text{BC}.$$

Again (*constr.* and S. 26. 1.) the $\triangle$ *gbh*, GBH are equiangular, as are, also, the $\triangle$ *ich*, ICH;

$\therefore$ (E. 4. 6.) $hg : hb :: HG : HB$:
And (*constr.*) $hb : hc :: HB : HC$:
Also (E. 4. 6.) $hc : hi :: HC : HI$;
$\therefore$ (E. 22. 5.) $hg : hi :: HG : HI$:

But (*constr.*) $\overline{hg} = \overline{hi}$; $\therefore$ $\overline{HG} = \overline{HI}$; and it is manifest, also, from the construction, that the $\angle$ GHI $=$ $\angle$ *ghi*, of the square *fghi*; $\therefore$ the $\angle$ GHI is a right $\angle$.

Again, since (*constr.*) $bh : hc :: BH : HC$,
$\therefore$ (*comp.* and *div.*) $th : bh :: TH : BH$:

Lastly, (*constr.* and S. 26. 1.) the two $\triangle$ *tfh*, TFH, are equiangular, as are, also, the two $\triangle$ *bhg*, BHG;

$\therefore$ (E. 4. 6.) $fh : th :: FH : TH$:
And $th : bh :: TH : BH$;
Also (E. 4. 6.) $bh : hg :: BH : HG$;
$\therefore$ (E. 22. 5.) $fh : hg :: FH : HG$:

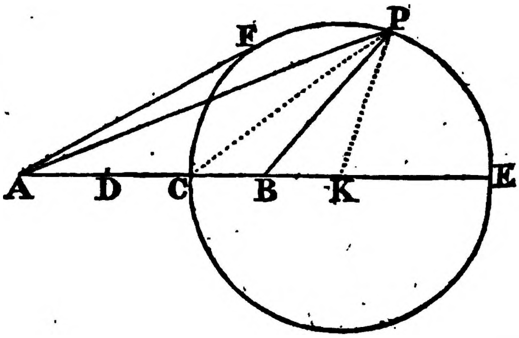
Wherefore, the two $\triangle$ *fhg*, FHG, having their sides about the equal $\angle$ *fhg*, FHG, proportionals, are (E. 4. 6.) equiangular; $\therefore$ the $\angle$ FGH is a right angle; and (E. 4. 6.) $\overline{FG} = \overline{GH}$, because (*constr.*) $\overline{fg} = \overline{gh}$: And, as hath been shewn, the $\angle$ FGH, GHI, are right $\angle$; $\therefore$ (E. 28. 1.) $\overline{GF}$ is parallel to $\overline{HI}$.

It has been shewn, also, that $\overline{HI} = \overline{HG}$; $\therefore$ (E. 33. 1. and E. 34. 1.) the figure FGHI is equilateral and rectangular: That is (E. 30. def. 1.) it is a square.

PROP. XXXIII.

41. **PROBLEM.** *To determine the locus of the summits of all the triangles which can be described on a given base, so that each of them shall have its two sides in a given ratio.*

Let AB be a given finite straight line: It is required to determine the *locus* of the summits of all the △s which can be described upon $\overline{AB}$, as a base, having their two remaining sides, in each, in a given ratio to one another.



Divide (E. 10. 6.) $\overline{AB}$ in C, so that AC shall be to CB in the given ratio; from the greater segment AC, cut off $\overline{CD} = \overline{CB}$; find (E. 11. 6.) a third proportional to $\overline{AD}$ and $\overline{CB}$, and in AB, produced, make $\overline{BK}$ equal to it; from the centre K, at the distance KC, describe the circle CPE: The circumference CPE is the *locus* which was to be determined.

For, take any point P, in the circumference CPE, and draw $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, and $\overline{PK}$: Then since,

(*constr.*) AD : CB : : CB : BK,
∴ (E. 18. 5.) AD + CB or AC : CB : : CK : BK;
∴ (E. 16. 5.) AC : CK : : CB : BK;
∴ (E. 18. 5.) AK : CK : : CK : BK;
i. e. (E. 15. def. 1.) AK : KP : : KP : KB;
∴ (E. 6. 6.) the two △ APK, BPK, are equiangular:
∴ (E. 4. 6.) PA : PB : : AK : PK or CK:
And it has been shewn that
AK : CK : : CK : BK : : AC : CB;
∴ (E. 11. 5.) PA : PB : : AC : CB:

And (*constr.*) AC is to CB in the given ratio; ∴ PA is to PB in the given ratio, wherever, in the circumference CPE, the point P is taken.*

PROP. XXXIV.

42. PROBLEM. *The base, the perpendicular distance of the vertex from the base, and the ratio of the two sides of a triangle being given, to construct it.*

Draw (E. 31. 1. and E. 11. 1.) a straight line parallel to the given base, and at a perpendicular distance from it equal to the given perpendicular distance; draw, (S. 33. 6.) the locus of the summits of all the △ which can be described on the given base, having their sides to one another in the given ratio; and it is manifest that the point, in which this locus meets the line drawn parallel to the base, will be the summit of the △ which was to be described.

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* If the given ratio be a ratio of *equality*, the locus to be determined is, manifestly, the straight line drawn at right angles to AB, through the point which divides AB into two equal parts.

Prop. XXXV.

43. Problem. *The segments into which the perpendicular, drawn from the vertex to the base of a triangle, divides the base, and the ratio of the two remaining sides being given, to construct the triangle.*

The segments being placed in the same straight line, upon their aggregate draw (S. 33. 6.) the *locus* of the summits of all the △, which can be described on that line, as a base, so as to have their remaining sides in the given ratio: And it is evident that a perpendicular drawn (E. 11. 1.) to this base, from the point, which is common to the two segments, will cut the *locus* in a point, which is the vertex of the △ that was to be described.

Prop. XXXVI.

44. Problem. *To find a point, from which if three straight lines be drawn to three given points, they shall be each to each in given ratios.*

Upon the straight line joining two of the given points, describe (S. 33. 6.) the *locus* of the summits of all △ having that line for a base, and having their sides to one another in one of the given ratios; upon the straight line, also, joining the third given point, and either of the other two, describe the *locus* of the summits of all △ having that line for a base, and having their sides in another of the given ratios: Then it is manifest, that the point, in which the one *locus* cuts the other, is the point which was to be found.

PROP. XXXVII.

45. **PROBLEM.** *A straight line being divided into three given parts, to find a point without it, at which the three parts shall subtend equal angles.*

Upon the aggregate of the first and second of the given parts, describe (S. 33. 6.) the *locus* of the summits of all △ having that line for a base, and having their sides to one another, as the first is to the second of the given parts: Again, upon the aggregate of the second and third of the given parts, describe the *locus* of the summits of all △ having that line for a base, and having their sides to one another as the second of the given parts is to the third: Then it is manifest, from E. 3. 6., that the point, in which the one *locus* cuts the other, is the point which was to be found.

PROP. XXXVIII.

46. PROBLEM. *To find a point in a given line, from which, if two straight lines be drawn to two given points, both on the same side of the given line, they shall be to each other in a given ratio.*

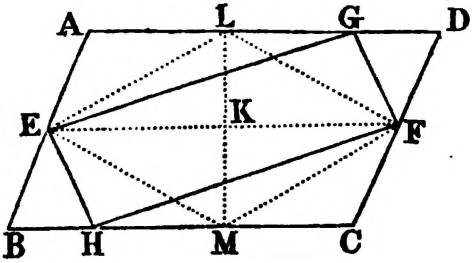
Upon the straight line joining the two given points, describe (S. 33. 6.) the *locus* of the summits of all △ having that line for a base, and having their sides in the given ratio; and it is evident, that the point, in which the locus, so described, cuts the given line, is the point which was to be found.

PROP. XXXIX.

47. PROBLEM. *In a given parallelogram to inscribe a parallelogram that shall have its two adjacent sides in a given ratio to one another, and that shall be the half of the given parallelogram.*

Let ABCD be the given ▭: It is required to inscribe in it a ▭, which shall be the half of ABCD, and which shall have two adjacent sides in a given ratio to one another.

Bisect (E. 10. 1.) AB in E, and through E draw (E. 31. 1.) $\overline{EF}$ parallel to $\overline{AD}$ or $\overline{BC}$: And, first,



if the given ratio be a ratio of equality, bisect, also, EF in K; through K draw (E. 11. 1.) $\overline{LKM}$ ⊥ to $\overline{EF}$; and draw $\overline{EL}$, $\overline{LF}$, $\overline{FM}$, and $\overline{ME}$: Then ELFM is an equilateral ▭, and it is the half of the ▭ ABCD.

For (E. 10. 6.) $\overline{LM}$ is divided, in K, in the same manner as $\overline{AB}$ is divided in E; ∴ $\overline{KL} = \overline{KM}$; ∴ (*constr.* and E. 4. 1.) $\overline{EL}$, and $\overline{LF}$, and $\overline{FM}$, and $\overline{ME}$, are equal to one another; and ∴ (S. 18. 1.) the figure LEMF is a ▭: And since (E. 4. 1.) the △ ELF is the half of the ▭ AEFD, and the △ EMF is the half of the ▭ EBCF, ∴ the whole figure ELFM is the half of the given ▭ ABCD.

But, secondly, let the given ratio be not a ratio of equality: In this case, upon EF describe (S. 33. 6.) the *locus* of all the △ having EF for a base, and having their sides in the given ratio, and let it cut AD in G; join E, G, and F, G; from CB cut off CH = AG, and join E, H and F, H: Then is EGFH the ▭ which was to be described.

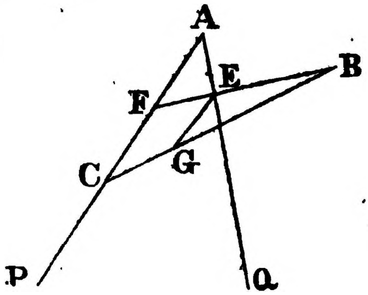
For (*constr.* and S. 43. 1.) EGFH is a ▭; and it may be shewn to be the half of the given ▭

ABCD, in the same manner as ELFM was shewn to be the half of ABCD; and (*constr.*) the adjacent sides EG and GF, are to one another in the given ratio.

Prop. XL.

48. **Problem.** *From a given point, either within or without a given rectilineal angle, to draw a straight line cutting the two lines which contain the angle, so that the distances of the two intersections from the given point, shall be to one another in a given ratio.*

Let PAQ be the given rectilineal ∠, and, first,



let B be a given point without it: It is required to draw from B a straight line cutting $\overline{AP}$ and $\overline{AQ}$, so that the distances of its intersections with $\overline{AP}$ and $\overline{AQ}$, from B, shall be to one another in a given ratio.

Through B draw BC to any point C in AP; find (E. 12. 6.) a fourth proportional to the two straight lines, which exhibit the given ratio, and to BC; and from BC cut off BG equal to that fourth proportional; through G draw (E. 31. 1.) GE parallel to AC, and meeting AQ in E; join B, E and produce it to F: Then shall FB be to EB in the given ratio.

For (*constr.* and E. 29. 1.) the two △ BFC, BEG, are equiangular:

∴ (E. 4. 6.) FB : EB :: CB : GB :

But (*constr.*) CB is to GB in the given ratio; ∴ FB is to EB in the given ratio.

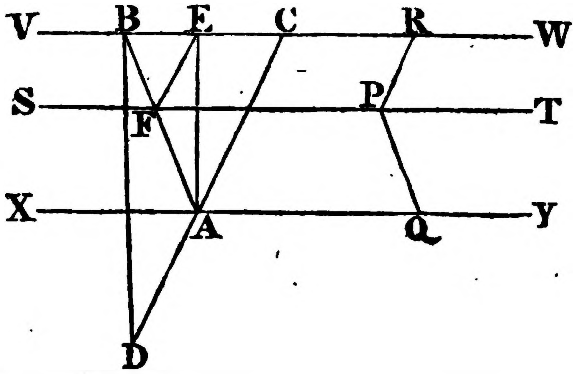
And, by the same method of construction, the problem may be solved, when the given point is within the given angle.

49. Cor. It is manifest that the problem admits of the same method of solution if one of the given lines, as AP, be a straight line of indefinite length, and if the other AQ be a line of any kind, in the same plane with AP.

Prop. XLI.

50. Problem. *To find, between two given parallel straight lines, the locus of all the points, from each of which if two straight lines be drawn to the two given parallels, so as always to make with them, towards the same parts, given angles, they shall be to one another in a given ratio.*

Let $\overline{VW}$ and $\overline{XY}$ be the two given parallels;



let $\overline{AB}$ and $\overline{AC}$, drawn from any point A in $\overline{XY}$, be in the two given directions: It is required to find, between $\overline{VW}$ and $\overline{XY}$, a *locus*, from any points of which if two straight lines be drawn to $\overline{VW}$ and $\overline{XY}$, the one parallel to $\overline{AC}$ and the other parallel to $\overline{AB}$, they shall be to one another in a given ratio.

Find (E. 12. 6.) a fourth proportional to the two straight lines, which exhibit the given ratio, and to $\overline{AB}$; and from $\overline{CA}$, produced, cut off AD equal to that fourth proportional; join B, D; through A draw (E. 31. 1.) $\overline{AE}$ parallel to $\overline{DB}$, and through E draw $\overline{EF}$ parallel to $\overline{CA}$, and let it meet $\overline{AB}$ in F; lastly, through F draw $\overline{SFT}$ parallel to $\overline{VW}$ or to $\overline{XY}$: Then is $\overline{ST}$ the *locus* which was to be found.

For take any point P, in $\overline{ST}$, and from P draw $\overline{PQ}$ parallel to $\overline{AB}$, and $\overline{PR}$ parallel to $\overline{AC}$.

And since (*constr.* and E. 29. 1.) the △s AEF, ABD are equiangular,

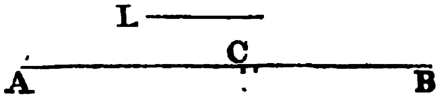
∴ (E. 4. 6.) FA : FE : : AB : AD :

But (*constr.* and E. 34. 1.) $\overline{FA} = \overline{PQ}$, and $\overline{FE} = \overline{PR}$; also (*constr.*) AB is to AD in the given ratio; ∴ PQ is to PR in the given ratio.

PROP. XLII.

51. **PROBLEM.** *To divide a given straight line into two parts, such, that the rectangle contained by the whole line and one of its parts, shall have a given ratio to the square of the other part.*

Let AB be the given straight line: It is required to divide it into two parts, such that the rectangle contained by AB and one of the parts shall have to the square of the other part a given ratio.

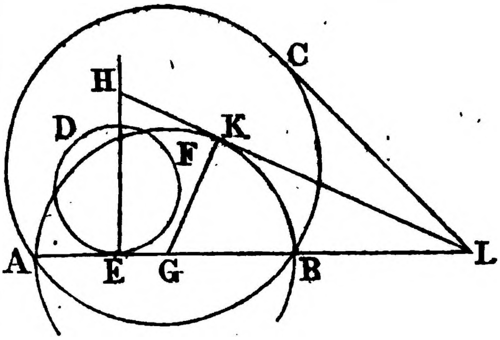


Find (E. 12. 6.) a fourth proportional, L, to the two straight lines, which exhibit the given ratio, and to AB; and divide (S. 81. 3.) AB into two parts, in C, so that $\overline{AC} \times L = \overline{CB}^2$: And since, (E. 1. 6.) $\overline{AC} \times \overline{AB} : \overline{AC} \times L$ or $\overline{CB}^2$: : AB : L, it is manifest that $\overline{AB}$ has been divided in C, so that $\overline{AC} \times \overline{AB}$ is to $\overline{CB}^2$ in the given ratio.

PROP. XLIII.

52. **PROBLEM.** *One given circle lying within another, to find a point from which, if two tangents be drawn, one to each of the given circles, they shall be to each other in a given ratio.*

Let ABC, DEF, be two given circles, of which DEF lies within ABC: It is required to find a point from which if tangents be drawn to touch the two circles ABC, DEF, they shall be to one another in a given ratio.



Draw (E. 17. 3.) $\overline{AL}$ touching the lesser circle DEF in any point E, and let $\overline{AL}$ meet the circumference of ABC in A and B; bisect (E. 10. 1.) AB in G, and from E draw (E. 11. 1.) $\overline{EH}$ $\perp$ to AB; find (E. 12. 6.) a fourth proportional to the two straight lines, which exhibit the given ratio, and to AG; and make EH equal to it; from the centre G, at the distance GA or GB, describe the circle AKB; from H draw (E. 17. 3.) $\overline{HK}$ touching the circle AKB in K; and produce $\overline{HK}$ to meet $\overline{AB}$, produced, in L: Then is L the point which was to be found.

For, from L draw $\overline{LC}$ touching the circle ABC in C; and join G, K; ∴ (*constr.* and E. 18. 3.) the ∠ GKL is a right ∠, as is, also, (*constr.*) the ∠ LEH; ∴ (S. 26. 1.) the △s LKG, LEH, are equiangular;

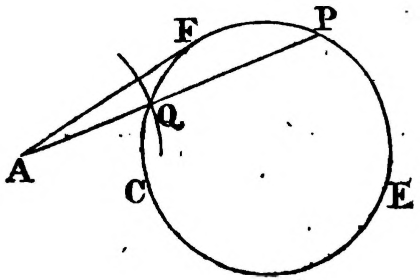
∴ (E. 4. 6.) LE : LK : : EH : GK or GA:

But, since (E. 36. 3.) $\overline{AL} \times \overline{LB}$ is equal to $\overline{LK}^2$, and also to $\overline{LC}^2$, ∴ $\overline{LK} = \overline{LC}$; and (*constr.*) EH is to GA in the given ratio; ∴ the tangent LE is to the tangent LC in the given ratio.

PROP. XLIV.

53. PROBLEM. *From a given point, to draw a straight line to cut a given circle, so that the distances of the two intersections from the given point, shall be to each other in a given ratio.*

Let CFE be the given circle, and A the given point without it: It is required to draw from A a straight line cutting CFE, so that the distances



of its two intersections from A shall be to one another in a given ratio.

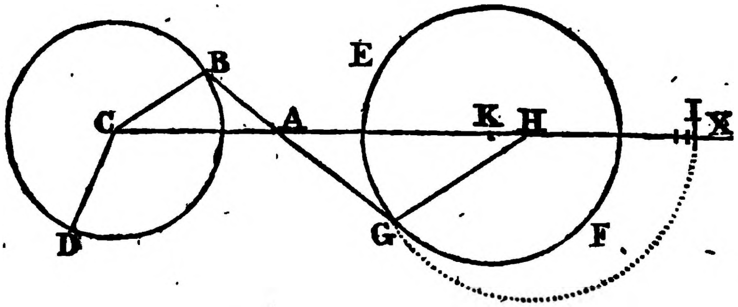
From A draw (E. 17. 3.) $\overline{AF}$ touching the circle CFE in F; find (S. 21. 6.) a square, which shall be to the square of AF, in the given ratio; from the centre A, at a distance equal to the side of the square thus found, describe a circle cutting the circumference of CFE in Q; and draw $\overline{AQ}$, which is, ∴, equal to the side of that square; produce AQ to meet the circumference of CFE again in P: Then shall AP be to AQ in the given ratio.

For (E. 1. 6.) $AP : AQ :: \overline{AP} \times \overline{AQ} : \overline{AQ}^2$: But (E. 36. 3.) $\overline{AP} \times \overline{AQ} = \overline{AF}^2$; and (*constr.*) $\overline{AF}^2$ is to $\overline{AQ}^2$ in the given ratio; ∴ $\overline{AP}$ is to $\overline{AQ}$ in the given ratio.

PROP. XLV.

54. **PROBLEM.** *Two given circles lying wholly without one another, through a given point, which is between the two circles, and which is posited in the straight line joining their centres, to draw a straight line that shall be terminated by the convex circumferences, and divided, by the given point, into two parts, that are to one another in a given ratio.*

Let BD and EF be two given circles, and A a given point in $\overline{CK}$, which joins the two centres C and K: It is required to draw, through A, a straight line, which being terminated by the convex circumferences of the circles BD and EF, shall be divided by A into two parts, that are to one another in a given ratio.



Produce $\overline{CK}$ indefinitely toward X: Find (E. 12. 6) a fourth proportional to the two lines, which exhibit the given ratio, and to CA, and from $\overline{AX}$ cut off $\overline{AH}$ equal to it; find, also, a fourth proportional to the same two given lines and any semi-diameter, CD, of the circle BD; and from $\overline{HX}$ cut off $\overline{HI}$ equal to it; from the centre H, at the distance HI, describe a circle, and let it cut the circumference of EF in G; draw $\overline{HG}$, which ∴ is equal to HI; draw (E. 31. 1.) $\overline{CB}$ parallel to HG; and join B, A, and G, A: Then shall BA and AG be in the same straight line;

for (*constr.* and E. 11. 5.) CA : AH : : CB : GH;
∴ (E. 16. 5.) CA : CB : : AH : GH;

and (*constr.* and E. 29. 1.) the ∠ BCA = ∠ AHG, and two remaining ∠∠ BAC, HAG, of the △ CBA, AGH, are of the same species, each of them being, necessarily, less than a right ∠; ∴ (E. 7. 6.) the ∠ BAC = ∠ GAH; ∴ BA and AG are in the same straight line; otherwise (E. 15. 1.) the greater of two ∠∠ would be equal to the less: And since (E. 7. 6.) the two △ CBA, AGH, are equiangular,

∴ (E. 4. 6.) BA : AG : : CA : AH;

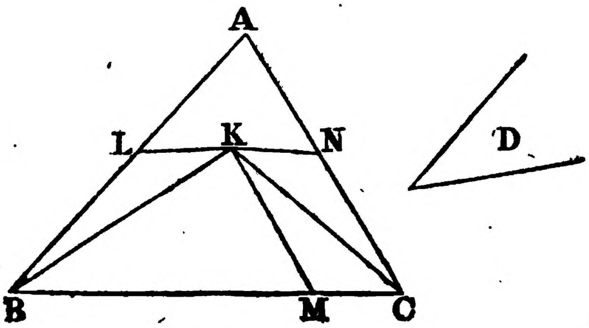
that is (*constr.*) BA is to AG in the given ratio.

Prop. XLVI.

85. Problem. *To find a point, from which if three straight lines be drawn to meet as many given straight lines, which cut one another, so as*

to make, each with the line on which it falls, an angle equal to a given angle, the lines so drawn shall be, each to each, in given ratios.

Let AB, BC, and CA, be the three given



straight lines, and D the given ∠: It is required to find a point from which if three straight lines be drawn to AB, BC, and CA, each making with each an ∠ equal to the ∠ D, they shall be to one another in given ratios.

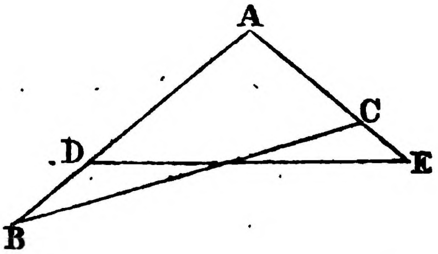
Straight lines being supposed to be drawn at the point B making, with AB and BC, angles each equal to the ∠ D, draw (S. 7. 6.) the locus BK of all the points from which if parallels be drawn to them meeting AB and BC, these parallels shall be to one another in the first of the given ratios; then (F. 29. 1.) shall the parallels so drawn make with AB and BC, angles each equal to the given ∠ D.

In like manner draw the locus CK of all the points from which if straight lines be drawn to AC and CB, making with them angles equal each to the given ∠ D, they shall be to one another in the second of the given ratios: And let BK and CK meet in K. It is manifest that K is the point which was to be found.

Prop. XLVII.

56. Problem. *To make an isosceles triangle, which shall be equal to a scalene triangle, and shall also have an equal vertical angle with it.*

Let ABC be the given scalene triangle: It is



required to make an equal isosceles △, which shall have the ∠ BAC for its vertical angle.

Find (E. 13. 6.) a mean proportional between the two unequal sides AB and AC, of the given

△ ABC, and from AB, the greater side, cut off AD equal to the mean proportional so found; also produce AC to E, so that $AE = AD$, and join D, E: Then is ADE the △ which was to be described.

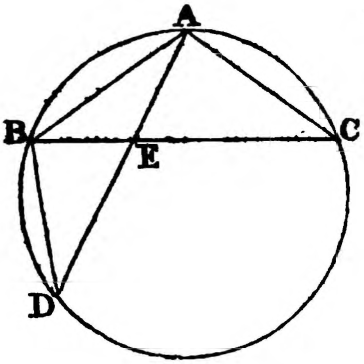
For (*constr.*) $BA : AE :: AD : AC$; ∴ (E. 15. 6.) the isosceles △ ADE is equal to the given △ ABC.

Prop. XLVIII.

57. **Theorem.** *If a straight line, drawn from the vertex of an isosceles triangle cutting the base, be produced to meet the circumference of a circle described about the triangle, the rectangle contained by the whole line so produced, and the part of it between the vertex and the base, shall be equal to the square of either of the equal sides of the triangle.*

Let $\overline{AD}$ drawn from the vertex, A, of the isosceles △ ABC, inscribed in the circle ABDC, cut the base of the △ in E; and the circumference of the circle in D: Then $\overline{DA} \times \overline{AE} = \overline{AB}^2$.

For join B, D, and since (*hyp.*) $\overline{AB} = \overline{AC}$, ∴ (E. 28. 3.) $\widehat{AB} = \widehat{AC}$, and ∴ (E. 27. 3.) the ∠



BDA, of the △ ABD, is equal to the ∠ ABE, of the △ AEB; and the ∠ BAD is common to the two △s; ∴ (S. 26. 1.) they are equiangular;

∴ (E. 4. 6.) DA : AB : : AB : AE;

∴ (E. 17. 6.) $\overline{DA} \times \overline{AE} = \overline{AB}^2$.

Prop. XLIX.

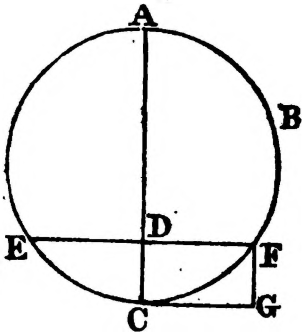
58. Theorem. *If from a given point, without a circle, two straight lines be drawn to the concave circumference, they shall be reciprocally proportional to the parts of them between the given point and the convex circumference.*

For (E. 36. 3. *cor.*) the rectangle contained by the one of the lines, so drawn, and the part of it without the circle, is equal to the rectangle contained by the other line and the part of it without the circle; ∴ (E. 16. 6.) the two straight lines so drawn are reciprocally proportional to the parts of them, between the given point and the convex circumference.

PROP. L.

59. PROBLEM. *To divide a given finite straight line into two parts, such, that another given straight line, not greater than the half of the former, shall be a mean proportional between them.*

Let AC be the given straight line which is to



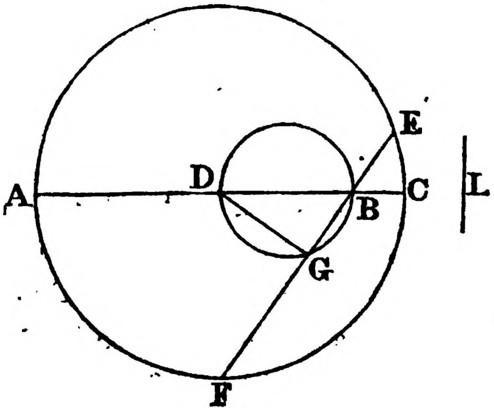
be divided into two parts, and let CG, placed at right ∠ to AC, be the line which is to be a mean proportional between the parts of ACD.

Upon $\overline{AC}$, as a diameter, describe the circle ACB; through G draw (E. 31. 1.) $\overline{GF}$ parallel to $\overline{CA}$, and let $\overline{GF}$ meet the circumference of AECF in F; through F draw the chord FDE parallel to GC, and ∴ (E. 29. 1.) $\perp$ to $\overline{AC}$; ∴ (E. 3. 3.) $\overline{FE}$ is bisected in D; ∴ (E. 35. 3.) $\overline{AD} \times DC = \overline{DF}^2$; but (*constr.* and E. 34. 1.) $\overline{CG} = \overline{DF}$; ∴ $\overline{AD} \times \overline{DC} = \overline{CG}^2$; ∴ (E. 17. 6.) the given straight line CG is a mean proportional between $\overline{AD}$ and $\overline{DC}$.

Prop. LI.

60. **Problem.** *Of four straight lines which are continual proportionals, the two extremes being given, and also a line which is equal to the difference of the other two, to find those two lines.*

Let AB and BC, placed in the same straight



line, be the two given extremes, and L the given difference of the two mean terms, of four proportionals: It is required to determine the two mean terms.

Bisect (E. 10. 1.) AC in D, and from the centre D, at the distance DA or DC, describe the circle AECF; likewise, upon DB, as a diameter, describe the circle DGB; and, since (S. 4. 5. *cor.* and *hyp.*) DB is greater than L, in the circle DGB place (E. 1. 4.) BG equal to the half of L; and produce GB both ways to meet the circumference in E and F: Then are BE and BF the two mean proportionals, which were to be found.

For join D, G; and because the ∠ DGB is in a semicircle, it is (E. 31. 3.) a right ∠; ∴ (E. 3. 3.) GF = GE; whence it is manifest that BG, which was made equal to the half of L, is the difference between BF and BE; also (E. 35. 3.) $\overline{AB} \times \overline{BC} = \overline{BF} \times \overline{BE}$;

∴ (E. 16. 6.) AB : BF : : BE : BC.*

Prop. LII.

61. Problem. *To make a triangle, which shall have its two sides equal to two given straight lines, each to each, and shall have its base equal*

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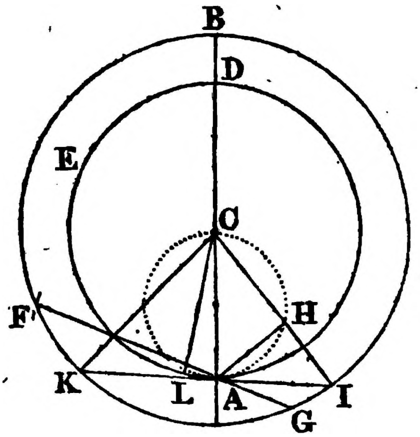
* The method used in this proposition furnishes another, and perhaps a neater, mode of solving the problems contained in S. 86. 3. and its corollary.

to the perpendicular distance of the vertex from the base.

Let AC and CB be two given straight lines: It



is required to describe a △ which shall have its base equal to the ⊥ drawn to it from the vertex, and shall have its two remaining sides equal to AC and CB, each to each.

Let AC and CB be placed in the same straight line; and from the centre C, at the distances CA and CB, describe the circles ADE, BFG; from the centre A, at the distance CB, describe a circle cutting the circumference BFG in F; and join A, F; so that AF=CB; produce FA to meet the circumference BFG again in G; upon AC as a diameter describe the circle AHC, and in it place AH=AG; draw $\overline{CH}$ and produce it to meet the circumference BFG in I; lastly, join

I, A, and produce $\overline{IA}$ to meet the circumference BFG in K: Then is CAK the △ which was to be described.

For, draw (E. 12. 1.) CL ⊥ to AK; ∴ (E. 31. 3. and S. 26. 1.) the △s CLI, AHI, are equiangular; and since (E. 35. 3.) $\overline{AI} \times \overline{AK} = \overline{AG} \times \overline{AF}$, *i. e.* (*constr.*) $\overline{AI} \times \overline{AK} = \overline{AH} \times \overline{CK}$,

∴ (E. 16. 6.) AI : AH :: CK : AK:
But (E. 4. 6.) AI : AH :: CI or CK : CL;
∴ (E. 9. 5.) AK = CL:

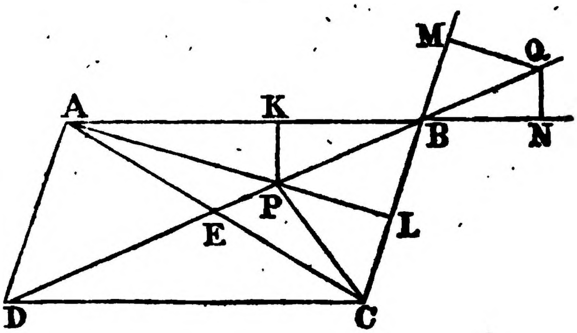
And the given straight line AC is one of the sides of the △ CAK; and CK, which (E. 15. def. 1.) is equal to CB, is the remaining side.

PROP. LIII.

62. THEOREM. *If from any point in the diameter, or the diameter produced, of a given parallelogram, perpendiculars be let fall on the two adjacent sides, produced, if necessary, which meet the diameter, the perpendiculars shall be reciprocally proportional to the sides on which they fall.*

Let AB be the diameter of the ▭ ABCD; let P be any point in AB, and Q any point in AB produced; and let PK and QM be ⊥ to AB, and PL and QN ⊥ to BC: Then

AB : BC :: PL : PK :: QM : QN.



For draw $\overline{AC}$, and $\overline{AP}$, and $\overline{PC}$; and let AC cut BD in E;

∴ (S. 42. 1.) $AE = EC$.

∴ (E. 38. 1.)

$\triangle ABE = \triangle CBE$; and $\triangle APE = \triangle CPE$;

∴ $\triangle APB = \triangle CPB$;

∴ (E. 41. 1.) $\overline{AB} \times \overline{PK} = \overline{BC} \times \overline{PL}$;

∴ (E. 16. 6.) $AB : BC :: PL : PK$.

Again, since (*constr.* E. 15. 1. E. 32. 1.) the $\triangle$s BKP, BNQ, are equiangular, as are, also, the $\triangle$s BLP, BMQ,

∴ (E. 4. 6.) $QM : BQ :: PL : PB$;

and $BQ : QN :: PB : PK$,

∴ (E. 22. 5.) $QM : QN :: PL : PK$:

And it has been proved that $AB : BC :: PL : PK$;

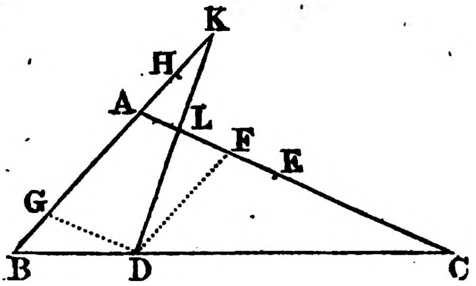
∴ (E. 11. 5.) $AB : BC :: QN : QM$.

In the same manner, also, the proposition may be shewn to be true, if perpendiculars be let fall from Q on the sides DA, and DC, produced.

PROP. LIV.

63. PROBLEM. *From a given point, in the base of a scalene triangle, to draw a straight line, which shall cut off equal segments from the two remaining sides, the less of those sides having been produced.*

Let D be a given point in the base BC of the scalene △ ABC: It is required to draw from D a straight line which shall cut off from the greater side AC, and from the less side AB, produced, equal segments.



From AC cut off (E. 3. 1.) CE = AB; through D draw (E. 31. 1.) DF parallel to AB, and DG parallel to AC; and, accordingly as the point F falls between E and C, or between E and A, take in AG, or in GA produced, AH = EF; produce HG, or GH, (S. 73. 3. *cor.*) to K, so that $\overline{GK} \times \overline{KH} = \overline{GA} \times \overline{AF}$; lastly, join D, K: Then shall DK cut off from AC a segment CL equal to the segment BK, which it cuts off from AB produced.

For, since (*constr.*) $\overline{GK}\times\overline{KH}$ is equal to $\overline{GA}\times\overline{AF}$, or (*constr.* and E. 34. 1.) to $\overline{DF}\times\overline{GD}$,

∴ (E. 16. 6.) GK : GD :: DF : KH :

But (*constr.* E. 29. 1. and S. 26. 1.) the two △ KGD, LFD, are equiangular;

∴ (E. 4. 6.) GK : GD :: DF : FL :

∴ (E. 9. 5.) KH = FL :

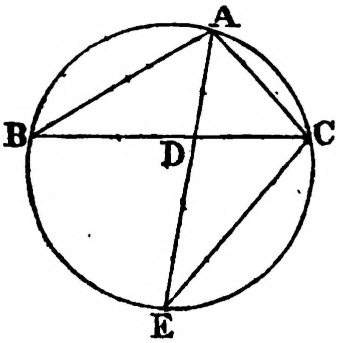
But (*constr.*) BH = CF; to these equals add the equals HK, and FL, and it is manifest, that the segment CL is equal to the segment BK.

PROP. LV.

64. THEOREM. *If an angle of a triangle be bisected by a straight line, which also cuts the base, the rectangle, contained by the sides of the triangle, is equal to the rectangle contained by the segments of the base, together with the square of the straight line bisecting the angle.*

Let ABC be a △, and let the ∠ BAC be bisected by $\overline{AD}$; then $\overline{BA}\times\overline{AC}=\overline{BD}\times\overline{DC}+\overline{AD}^2$.

Describe (E. 5. 4.) the circle ACB about the triangle; produce AD to the circumference in E, and draw $\overline{EC}$. And, because (E. 21. 3.) the ∠ ABC = ∠ AEC, and (*hyp.*) the ∠ BAD = ∠ CAE, ∴ (E. 32. 1.) the △ ABD, AEC, are equiangular;



∴ (E. 4. 6.) BA : AD :: EA : AC;

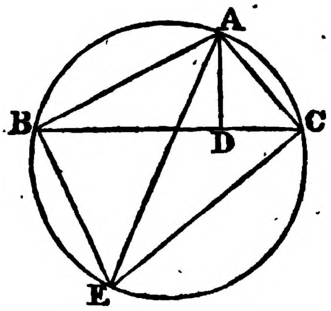
∴ (E. 16. 6.) $\overline{BA} \times \overline{AC} = \overline{EA} \times \overline{AD}$; *i. e.* (E. 3. 2.) $\overline{BA} \times \overline{AC} = \overline{ED} \times \overline{DA} + \overline{AD}^2$: But (E. 35. 3.) $\overline{ED} \times \overline{DA} = \overline{BD} \times \overline{DC}$; ∴ $\overline{BA} \times \overline{AC} = \overline{BD} \times \overline{DC} + \overline{AD}^2$.

PROP. LVI.

65. THEOREM. *If from any angle of a triangle a straight line be drawn perpendicular to the base, the rectangle contained by the sides of the triangle, is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.*

Let ABC be a △, and $\overline{AD}$ the ⊥ from the ∠ BAC to the base BC; then is $\overline{BA} \times \overline{AC}$ equal to the rectangle contained by AD, and the diameter of the circle described about the △ ABC.

Describe (E. 5. 4.) the circle ACB about the



triangle; draw its diameter AE, and join E, C: Because the ∠ ECA in a semi-circle is equal (E. 31. 3.) to the right ∠ BDA, and that (E. 21. 3.) the ∠ AEC = ∠ ABC, ∴ (E. 32. 1.) the △ ABD, AEC are equiangular,

∴ (E. 4. 6.) BA : AD :: EA : AC;

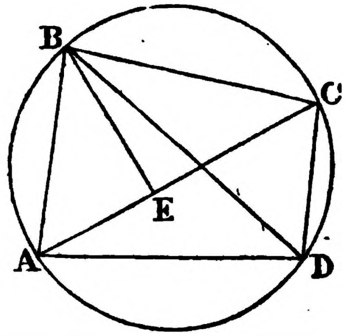
∴ (E. 16. 6.) $\overline{BA} \times \overline{AC} = \overline{EA} \times \overline{AD}$.

PROP. LVII.

66. **THEOREM.** *The rectangle contained by the diagonals of a quadrilateral rectilineal figure, inscribed in a circle, is equal to both the rectangles contained by its opposite sides.*

Let ABCD be any quadrilateral rectilineal figure, inscribed in a circle ACB, and let AC, BD, be its diagonals; then $\overline{AC} \times \overline{BD} = \overline{AB} \times \overline{CD} + \overline{AD} \times \overline{BC}$.

Make (E. 23. 1.) the ∠ ABE = ∠ DBC; add to each the common ∠ EBD; ∴ the ∠ ABD =



∠ EBC; and (E. 21. 3.) the ∠ BDA = ∠ BCE; ∴ (E. 32. 1.) the △s ABD, BCE, are equiangular;

∴ (E. 4. 6.) BC : CE :: BD : DA;

∴ (E. 16. 6.) $\overline{BC} \times \overline{AD} = \overline{BD} \times \overline{CE}$: Again, because (*constr.*) the ∠ ABE = ∠ DBC, and (E. 21. 3.) the ∠ BAE = ∠ BDC, the △s ABE, BCD, are equiangular;

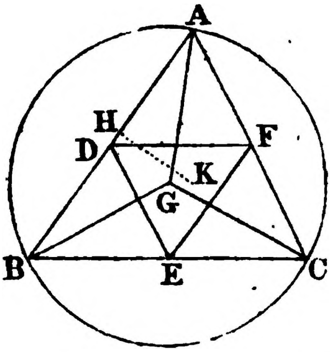
∴ (E. 4. 6.) BA : AE :: BD : DC;

∴ (E. 16. 6.) $\overline{BA} \times \overline{DC} = \overline{BD} \times \overline{AE}$: And it has been shewn that $\overline{BC} \times \overline{AD} = \overline{BD} \times \overline{CE}$; ∴ (E. 1. 2.) $\overline{AC} \times \overline{BD} = \overline{AB} \times \overline{CD} + \overline{AD} \times \overline{BC}$.

Prop. LVIII.

67. Theorem. *If, from the centre of the circle, described about a given triangle, perpendiculars be drawn to the three sides, their aggregate shall be equal to the radius of the circumscribed circle, together with the radius of the circle inscribed in the given triangle.*

Let ABC be the given △; bisect (E. 10. 1.)



AB, BC, and AC in the points D, E, and F; and from D, E, and F draw (E. 11. 1.) DG ⊥ to AB, EG ⊥ to BC, and FG ⊥ to AC; then (S. 4. 1.) these perpendiculars meet in the same point G, which is the centre of the circle that can be described about the △ ABC; find, also, (E. 4. 4.) the centre K, and the semi-diameter KH, of the circle that can be inscribed in the △ ABC; and draw GA: Then* GD + GE + GF = GA + KH.

For draw $\overline{DE}$, $\overline{EF}$, and $\overline{FD}$, ∴ (S. 69. 1. *cor.* 1. and E. 34. 1.) ½ AC, CF = ½ AB, and FD = × ½ BC; draw $\overline{GB}$, and $\overline{GC}$; And, since (*constr.*) the ∠∠ at D, E, F, are right ∠∠, ∴ (E. 32. 1. *cor.* 1.) the two ∠∠ DAF, DGF, are, together, equal to two right ∠∠; ∴ (S. 28. 3.) a circle may be described about the trapezium ADGF; and in the same manner it may be shewn that circles may be described about BDGE, and CFGE:

∴ (S. 57. 6.) $\overline{AG}\times\overline{DF}+\overline{BG}\times\overline{DE}+\overline{CG}\times\overline{FE}=$

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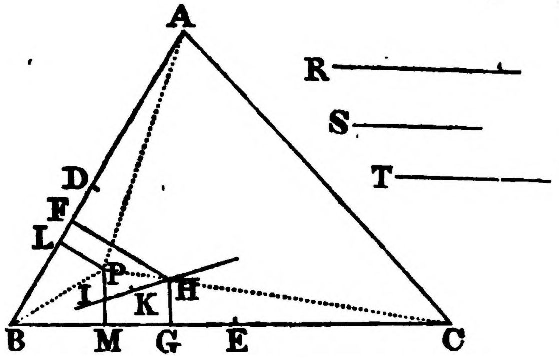
* The straight lines GD, GE, GF, are wanting in the figure.

$\overline{AF}\times\overline{DG}+\overline{AD}\times\overline{GF}+\overline{BD}\times\overline{GE}+\overline{BE}\times\overline{DG}+\overline{CE}\times\overline{GF}+\overline{CF}\times\overline{GE}$: And if to the doubles of these equals be added the rectangles $\overline{GE}\times\overline{BC}+\overline{GF}\times\overline{AC}+\overline{GD}\times\overline{AB}$, which (F. 41. 1.) make up the double of the △ ABC, it will be manifest, from E. 1. 2., that the rectangle contained by the perimeter of the △ ABC, and by GA, together with the double of the △ ABC, is equal to the rectangle contained by the perimeter of ABC, and by the aggregate of GD, GE, and GF: But (S. 2. 4.) the double of the △ ABC is equal to the rectangle contained by the perimeter of the △ and the semi-diameter, KH, of the circle inscribed in it; ∴ (E. 1. 2.) the rectangle contained by the perimeter, and by the aggregate of GA and KH, is equal to the rectangle contained by the perimeter, and by the aggregate of GD, GE, and GF; ∴ $GD+GE+GF=GA+KH$.

PROP. LIX.

68. PROBLEM. *To find a point, from which if three straight lines be drawn to three given points, their differences shall be severally equal to three given straight lines; the difference of any two of the straight lines to be drawn, not being greater than the distance of the two points to which they are to be drawn.*

Let A, B, C, be the three given points, and R, S,



two of the given differences: It is required to find a point, from which if three straight lines be drawn to A, B, and C, the difference of the first and second shall be equal to R, the difference between the second and third equal to S, and ∴ the difference between the first and third equal to the third of the given differences.

Draw $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$; bisect (E. 10. 1.) AB in D, and BC in E; from DB cut off DF, equal to a third proportional (E. 11. 6.) to 2AB, and to S; likewise from EB cut off EG, equal to a third proportional to 2BC, and to R; and through F and G draw (E. 11. 1.) FH ⊥ to AB, and GH ⊥ to BC, and let them meet in H; find (E. 12. 6.) a fourth proportional, (T,) to $\overline{AB}$, S, and BC; through H draw (S. 7. 6.) the *locus*, IH, of all the points, from which if perpendiculars be drawn to AB and BC, respectively, they shall cut off from GB and FB segments that are to one another as R is to T; lastly, in IH find (S. 96. 3.) a point K, such that the difference of its distances from C and B, shall be equal to R: Then is K the point which was to be found.

For if not, let P be the point; and, if it be possible, let the point P be out of IH; join P, A, and P, B, and P, C; and draw, from P (E. 12. 1.) PL $\perp$ to AB, and PM $\perp$ to BC: Then (*constr.* E. 17. 3. and S. 96. 3. *cor.* 1.)

$$\overline{FL}\times\overline{AB}=\overline{BP}\times\overline{S};\ \text{and}\ \overline{GM}\times\overline{BC}=\overline{BP}\times\overline{R};$$

∴ (E. 16. 6. and *constr.*)

BP : FL :: AB : S :: BC : T;

and GM : BP :: R : BC;

∴ (E. 23. 5.) GM : FL :: R : T:

∴ (*constr.* and S. 7. 6. *cor.*) the point P cannot be out of IH; ∴ (*constr.* and S. 96. 3. *cor.* 2.) K is the point which was to be found.

PROP. LX.

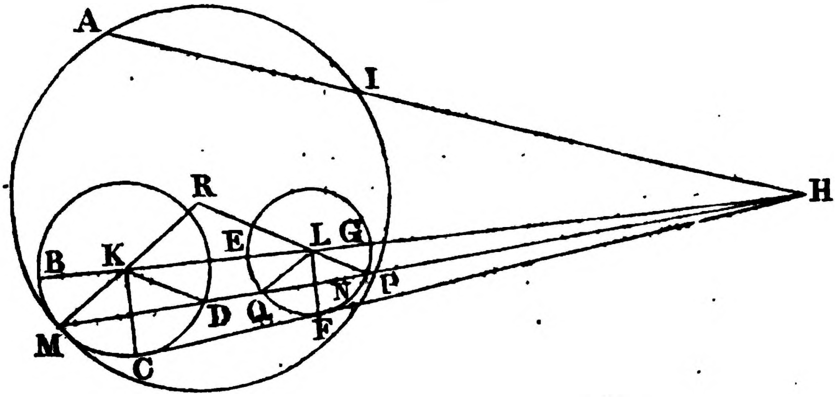
69. PROBLEM. *To describe a circle, which shall pass through a given point, and touch two given circles.*

Find a point (S. 59. 6.) such that the difference between its distance from the centre of the one circle, and its distance from the given point, shall be equal to the semi-diameter of that circle; and that the difference between its distance from the other centre, and from the given point, shall likewise be equal to the other given semi-diameter:

It is manifest (S. 6. 3.) that the point, so determined, is the point which was to be found.

Otherwise.

Let BCD and EFG be the two given circles,



and A a given point without them: It is required to describe a circle which shall pass through A, and touch the two circles BCD, EFG.

Find (E. 1. 3.) the centres, K and L, of the two given circles; draw $\overline{KL}$ and let it, produced, meet the circumference of BCD in B, and the circumference of EFG in E and G; and let it meet $\overline{CH}$, which is drawn (S. 52. 3.) so as to touch both the circles, in H; join H, A; find (E. 12. 6.) a fourth proportional to AH, HB, and HG, and from HA cut off HI equal to it; so that (E. 16. 6.) $\overline{BH} \times \overline{HG} = \overline{AH} \times \overline{HI}$; lastly, describe (S. 95. 3.) a circle AMI, passing through A and I, and touching either of the given circles BCD, in some point, M; it shall, also, if $\overline{HM}$ be drawn, pass through the point N, in which HM cuts the circumference of the circle EFG, and shall touch EFG in the point N.

For, if it be possible, let the circumference of the circle AMI cut HM in some other point, as P: Then (E. 36. 3.) $\overline{MH} \times \overline{HP} = \overline{AH} \times \overline{HI}$; but (*constr.*) $\overline{AH} \times \overline{HI} = \overline{BH} \times \overline{HG}$, and (E. 36. 3.) $\overline{BH} \times \overline{HG} = \overline{MH} \times \overline{HN}$; ∴ $\overline{MH} \times \overline{HP} = \overline{MH} \times \overline{HN}$; ∴ HP is equal to HN, the less to the greater, which is absurd; ∴ the circumference of the circle MIA cannot but meet the circle EFG in the point where it is cut by MH; and it touches the circle EFG in that point.

For draw KM, KC, KD, LQ, LF, and LP, and let MK and PL, produced, meet in R: Then since (*constr.* and E. 18. 3.) the △ HFL, HCK, having a common ∠ at H, have the ∠∠ HFL, HCK, right ∠∠, they are (S. 26. 1.) equiangular; ∴ (E. 4. 6.) HL:LF or LQ::HK:KC or KM: And the △ HLQ, HKM, have a common ∠ at H, and have the two remaining ∠∠ HQL, HMK of the same species; for since MH cuts both the circles, the ∠∠ HQL, HMK, are (E. 16. 3. *cor.*) each of them less than a right ∠; ∴ (E. 7. 6.) the ∠ HQL = ∠ HMK; ∴ (E. 15. def. 1. and E. 5. 1.) the ∠ LNQ, or RNM, = ∠ HMK, or NMR; ∴ (E. 6. 1.) RM = RN; but, since the circle AMI touches the circle BCD, of which K is the centre, ∴ (E. 11. or 12. 3.) the centre of AMI must be in MR; and since RM = RN, that centre (E. 7. 3.) must be in R; since, ∴, the diameters of the two circles MAI, EFG, have a common extremity at N, the two circles (S. 6. 3.) touch one another.[*]

Prop. LXI.

70. Problem. *To describe a circle that shall touch three given circles.*

Find a point (S. 59. 6.) such that the difference between its distances from the centres of the first and second of the given circles, shall be equal to the difference of the diameters of those circles, and such that the difference between its distances from the centres of the first and third of the given circles, shall be equal to the difference of the diameters of those circles: Then it is manifest, that the difference between its distances from the centres of the second and third of the given circles, will be equal to the difference of their diameters; and that, if from the point so determined, as a centre, a circle be described touching any one of the given circles, it will (S. 6. 3.) also touch the other two.

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[*] It is evident that Prop. 59 may be deduced from this proposition, as it is thus independently demonstrated; and that the proposition immediately following, which is one of some celebrity, may be deduced from either of them.

PROP. LXII.

71. PROBLEM. *Upon a given finite straight line, to describe an equilateral and equiangular figure, having the number of its sides equal to four, eight, sixteen, &c.; or to three, six, twelve, &c.; or to five, ten, twenty, &c.; or to fifteen, thirty, sixty, &c. sides.*

In any circle inscribe (S. 14. 4. *cor.* 3.) an equilateral and equiangular rectilineal figure of any number of sides that is specified in the proposition; then upon the given finite straight line describe (E. 18. 6.) a rectilineal figure similar to it, and the problem will have been solved.

PROP. LXIII.

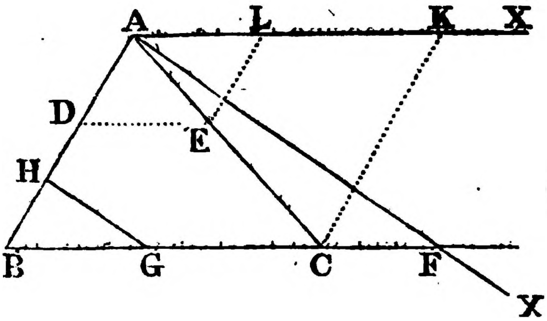
72. THEOREM. *Similar triangles, and similar polygons, are to one another as any rectilineal figure described upon any side of the one, is to a similar rectilineal figure similarly described upon the homologous side of the other.*

For (E. 20. 6.) the two given figures, and two similar figures thus similarly described, will have to one another the same duplicate ratio of that which the homologous sides have.

Prop. LXIV.

73. **Problem.** *To cut off from a given triangle any part required, by a straight line drawn parallel to a given straight line.*

Let ABC be the given Δ, and AX a given straight line: It is required to cut off, from the Δ ABC, any assigned part, by a straight line drawn parallel to AX.



First, let AX be parallel to BC; find (S. 21. 6.) a square which shall be the same part of the square of AB, that the Δ, to be cut off, is required to be of the given Δ, and make AD equal to its side; through D draw (E. 31. 1.) DE parallel to AX or BC: Then is ADE the Δ which was to be cut off from ABC.

For, (*constr.* and E. 29. 1.) the $\triangle$s ADE, ABC, are equiangular;

∴ (E. 4. 6. and S. 63. 6.) $\overline{AD}^2:\overline{AB}^2::\triangle ADE:\triangle ABC$.

∴ (*constr.* and S. 4. 5.) the $\triangle$ ADE is the required part of the $\triangle$ ABC.

Secondly, let AX be not parallel to BC, and let it meet BC, produced, if necessary, in F: Find (S. 21. 6.) a square which shall be the same part of the rectangle $\overline{FB}\times\overline{BC}$, that the $\triangle$, to be cut off, is required to be of the $\triangle$ ABC, and make BG equal to its side; through G draw GH parallel to FA: Then is BHG the $\triangle$ which was to be cut off from ABC.

For (*constr.* and E. 29. 1.) the $\triangle$s BHG, BAF, are equiangular;

∴ (E. 4. 6. and S. 63. 6.)

$$\overline{BG}^2:\overline{BF}^2::\triangle BHG:\triangle BAF;$$

And (E. 1. 6. and E. 11. 5.)

$$\overline{BF}^2:\overline{BF}\times\overline{BC}::\triangle BAF:\triangle BAC;$$

∴ (E. 22. 5.)

$$\overline{BG}^2:\overline{BF}\times\overline{BC}::\triangle BHG:\triangle BAC;$$

∴ (*constr.* and S. 4. 5.) the $\triangle$ BHG is the required part of the $\triangle$ ABC.

Prop. LXV.

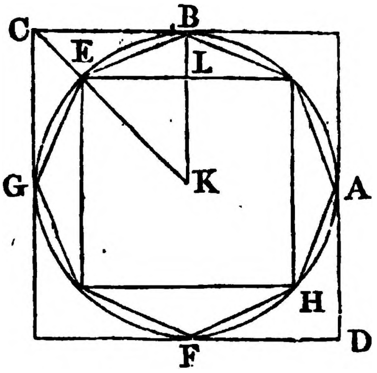
74. Problem. *To describe a polygon, similar to a given polygon, and having a given ratio to it.*

Upon any side of the given polygon describe (E. 46. 1.) a square; find (S. 21. 6.) a square which shall have to the square first described the given ratio; and upon its side describe (E. 18. 6.) a polygon similar, and similarly situated, to the given polygon: It is manifest, from E. 20. 6., that it will have to the given polygon the given ratio.

PROP. LXVI.

75. THEOREM. *Any regular polygon, inscribed in a circle, is a mean proportional between the inscribed and circumscribed regular polygons of half the number of sides.*

Let BGFA be a polygon inscribed in the circle BG, and let EH and CD be polygons of half the number of sides, the one EH inscribed in the



circle (S. 14. 4. *cor.* 3.) by joining the sides of the figure BGFA, and the other CD described about the circle, by drawing tangents to it through the angular points A, B, G, and F; so that (E. 18. 3. E. 28. 3. E. 27. 3. E. 26. 3. and S. 19. 3.) it is equilateral and equiangular: Then is the polygon BGFA a mean proportional between the polygons EH and CD.

Find (E. 1. 3.) the centre K of the circle BGFA, and join K, B, and K, C: It is manifest, from the construction, that KB bisects, at right ∠, the sides of the figures EH and CD, which it cuts, and that KC passes through the angular point E:

And (E. 1. 6. and E. 4. 6.)

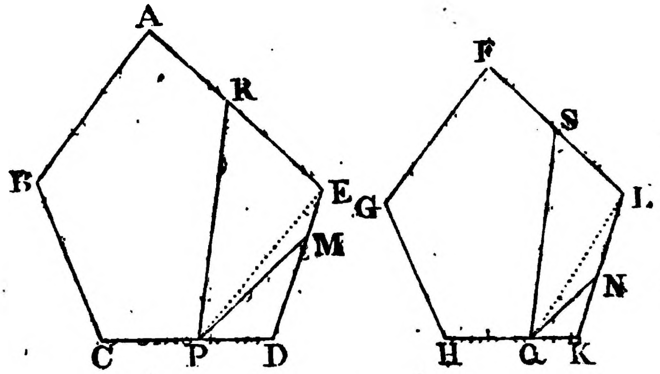
△ CBK: △ EBK :: CK:EK :: CB:EL:

But, the figure CD is the same multiple of the △ CBK, that the figure BGFA is of the △ EBK; also a side of CD is double of CB; and a side of EH is double of EL; ∴ (E. 15. 5.) CD is to BGFA as a side of CD is to a side of EH; and (E. 20. 6.) CD has to EH the duplicate ratio, of that which a side of CD has to a side of EH; ∴ CD has to EH the duplicate ratio, of that which it has to BGFA; *i. e.* (E. 10. def. 5.) the figure BGFA is a mean proportional between CD and EH.

PROP. LXVII.

76. THEOREM. *If from two points similarly situated, one in each of any two homologous sides of two similar polygons, two straight lines be drawn making equal angles with those sides, they shall cut off from the polygons two similar figures; and the one shall be the same part of the one polygon, that the other is of the other.*

Let AC and FH be two similar polygons, and P



and Q two points similarly situated in the two homologous sides CD and HK: If from P and Q straight lines be drawn, making equal ∠∠ with CD and HK, they shall cut off similar figures from the polygons; and the one shall be the same part of the one polygon that the other is of the other.

First, let PM and QN, making the ∠ MPD = ∠ NQK, cut the sides DE and KL, adjacent to

CD and HK: And since (*hyp.* and S. 26. 1.) the two △ MPD, NQK, are equiangular, they are (E. 4. 6.) similar to one another, and they are to one another (E. 19. 6.) in the duplicate ratio of their homologous sides PD and QK, that is (*hyp.*) in the duplicate ratio of CD and HK; ∴ (E. 20. 6.) they are to one another in the same ratio as the polygons are, and ∴ whatever part the △ MPD is of the polygon ABCDE, the same part is the △ NQK of the polygon FGHKL.

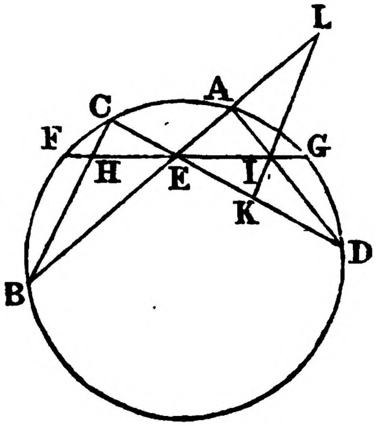
Secondly, let PR and QS cut any other sides of the polygons, as AE and FL, which are not adjacent to the sides CD and HK: Draw PE and QL; and since (*hyp.* and E. 26. 1.) the △ EPD and LQK are equiangular, the △ RPE and SQL are also equiangular; whence it may be shewn, (as in E. 20. 6.) that RPDE, SQKL, are similar figures; ∴ (E. 20. 6.) they are to one another in the duplicate ratio of the homologous sides DE and KL; or in the ratio of the polygon ABCDE to the polygon FGHKL; ∴ RPDE is the same part of ABCDE that SQKL is of FGHKL.

PROP. LXVIII.

77. THEOREM. *If any two chords of a circle intersect each other, the straight lines joining their extremities shall cut off equal segments from the chord which passes through the common inter-*

section of the two former chords and is there bisected.

Let AB and CD be two chords of the circle



ACBD, cutting one another in E; through E draw (S. 2. 3.) the chord FG, so that FG is bisected in E; and join C, B and A, D: Then shall HE = EI.

For through I draw (E. 31. 1.) KIL parallel to BC, and meeting CD in K, and BA, produced, in L: Then (*constr.* and E. 29. 1.) the ∠CBL = ∠ BLK, and that (E. 21. 3.) the ∠ CBA = ∠ CDA, ∴ the ∠ ALI = ∠IDK; and (E. 15. 1.) the ∠ AIL, of the △ LAI, is equal to the ∠KID, of the △ DKI; ∴ (S. 26. 1.) these two △s are equiangular, as are also (*constr.* and E. 29. 1.) the two △s CEH, IEK, and the two △s HEB, IEL;

∴ (E. 4. 6.) AI : IL :: KI : ID;

∴ (E. 16. 6.) $\overline{IL} \times \overline{KI} = \overline{AI} \times \overline{ID}$:

Again (E. 4. 6.) $CH : HE :: IK : IE$,
and $BH : HE :: IL : IE$,
∴ (E. 22. 6.)
$\overline{CH}\times\overline{BH} : \overline{HE}^2 :: \overline{IK}\times\overline{IL}$ or $\overline{AI}\times\overline{ID} : \overline{IE}^2$;
∴ (E. 18. 5.)
$\overline{CH}\times\overline{BH}+\overline{HE}^2 : \overline{HE}^2 :: \overline{AI}\times\overline{ID}+\overline{IE}^2 : \overline{IE}^2$;
∴ (E. 35. 3.)
$\overline{FH}\times\overline{HG}+\overline{HE}^2 : \overline{HE}^2 :: \overline{FI}\times\overline{IG}+\overline{IE}^2 : \overline{IE}^2$:

But (*hyp.* and E. 5. 2.) $\overline{FH}\times\overline{HG}+\overline{HE}^2$, and $\overline{FI}\times\overline{IG}+\overline{IE}^2$, are each of them equal to $\overline{EF}^2$ or $\overline{EG}^2$, and ∴ they are equal to one another; ∴ (E. 14. 5.) $\overline{HE}^2=\overline{IE}^2$; ∴ $HE = IE$.

PROP. LXIX.

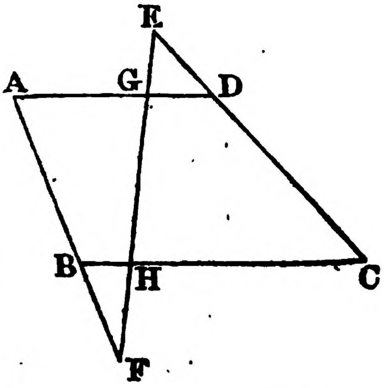
78. PROBLEM. *Two similar rectilineal figures being given, to find a third figure also similar to them and a mean proportional between them.*

Find (E. 13. 6.) a mean proportional between any two homologous sides of the given figures, and upon it describe (E. 18. 6.) a rectilineal figure similar to either of them, and ∴ (E. 21. 6.) similar, also, to the other: Then (E. 22. 6.) will the rectilineal figure, so described, be a mean proportional between the two given figures.

PROP. LXX.

79. PROBLEM. *If two sides of a trapezium be parallel, and a straight line be drawn cutting them, and meeting also the other two sides, (any of the sides being produced, if necessary) the two rectangles contained by the respective segments of the parallel sides, have to each other the same ratio, as the two rectangles contained by the segments into which the line, so drawn, is severally divided by each of the two parallels.*

Let the side AD, of the trapezium ABCD, be parallel to the opposite side BC, and let EF cut AD and BC, in G and H, and AB and DC, produced, in F and E: Then $\overline{AG} \times \overline{GD} : \overline{BH} \times \overline{HC} :: \overline{GF} \times \overline{EG} : \overline{HF} \times \overline{EH}$.



For (*hyp.* and E. 29. 1.) the △ AGF, BHF, and the △ EGD, EHC, are equiangular; ∴ (E. 4. 6.)

$$AG:BH::GF:HF;$$

$$\text{and } GD:HC::EG:EH;$$

∴ (S. 1. 6.)

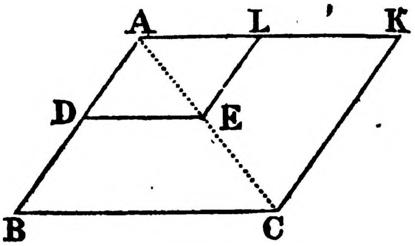
$$\overline{AG}\times\overline{GD}:\overline{BH}\times\overline{HC}::\overline{GF}\times\overline{EF}:\overline{HF}\times\overline{EH}.$$

Which conclusion may also be arrived at by means of E. 23. 6.

Prop. LXXI.

80. Problem. *To cut off from a given parallelogram a similar parallelogram, which shall be any required part of it.*

Let ABCK be the given ▱: It is required to



cut off from it a similar ▱, which shall be any required part of it.

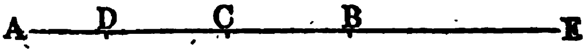
Draw the diameter AC, and from the △ ABC cut off (S. 64. 6.) by a straight line DE, drawn parallel to AK, the △ ADE the same part of ABC as the ▱ to be cut off is required to be of the given ▭; through E draw (E. 31. 1.) EL parallel to AB: Then, since (E. 34. 1.) the ▭ ADEL is the double of the △ ADE, it is (*constr.* and E. 15. 5.) the required part of the ▭ ABCK; and (E. 24. 6.) the ▭ ADEL is, also, similar to the ▭ ABCK.

81. Cor. Hence, a gnomon may be cut off from a given ▭, which shall be any required part of it.

Prop. LXXII.

82. Theorem. *A given straight line being cut in extreme and mean ratio, if from the greater segment the less be taken, the greater segment also will thus be cut in extreme and mean ratio; and if a straight line, equal to the greater segment, be added to the given line, the line which is made up of the given line and this segment, is also cut in extreme and mean ratio.*

Let AB be a given finite straight line; let it be



cut (E. 30. 6.) in extreme and mean ratio in the point C; from the greater segment, AC, cut off CD = CB; and to AB add BE = AC: Then shall AC be cut in extreme and mean ratio in the point D; and AE shall be cut in extreme and mean ratio in the point B.

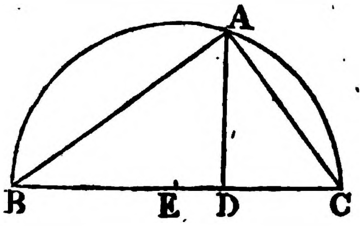
For since (*hyp.*) AB:AC :: AC:CB or CD,
∴ (E. 17. 5.) CB or CD:AC :: AD:CD;
∴ (S. 2. 5.) AC:CD :: CD:AD;
∴ (E. 3. def. 6.) AC is cut in extreme and mean ratio in the point D.

Again, since
(*hyp.* and S. 2. 5.) AC or BE:AB :: CB:AC,
∴ (E. 18. 5.) AE:AB :: AB:AC or BE;
∴ (E. 3. def. 6.) AE is cut in extreme and mean ratio in the point B.

Prop. LXXIII.

83. **Problem.** *Upon a given straight line, as an hypotenuse, to describe a right-angled triangle, which shall have its three sides continual proportionals.*

Let BC be the given finite straight line: It is required to describe upon it a right-angled △, the sides of which shall be continual proportionals.



Cut (E. 30. 6.) BC in extreme and mean ratio, in the point D; bisect (E. 10. 1.) BC in E;

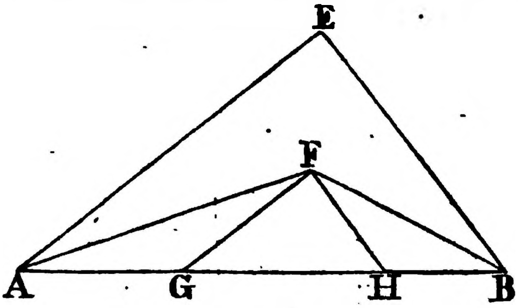
from the centre E, at the distance EB or EC, describe the circle BAC, and let DA, drawn from D, (E. 11. 1.) ⊥ to BC, meet its circumference in A, and join A, B and A, C: Then is ABC the △ which was to be described.

For (*constr.* and E. 31. 3.) the ∠ BAC is a right ∠; ∴ (*constr.* and E. 8. 6. *cor.*) AC is a mean proportional between BC and DC, as is also (*constr.*) BD; ∵. AC = BD; ∵. but (E. 8. 6. *cor.*) AB is a mean proportional between BC and BD; ∴ AB is a mean proportional between BC and AC.

Prop. LXXIV.

84. Problem. *The perimeter being given of a right-angled triangle, having its three sides proportionals, to construct the triangle.*

Let AB be a given straight line: It is required



to describe a right-angled △, which shall have its sides continual proportionals, and equal together to AB.

Upon AB describe (S. 73. 6.) the right-angled △ AEB, having its sides continual proportionals; bisect (E. 9. 1.) the ∠∠ EAB, EBA, by two straight lines AF and BF, which meet in F; and through F draw (E. 31. 1.) FG parallel to EA, and FH parallel to EB: Then is FGH the △ which was to be described.

For it may be shewn, as in S. 34. 1., that the perimeter of the △ FGH is equal to AB; and since (*constr.* E. 29. 1. and S. 26. 1.) the △△ FGH and EAB are equiangular, and that the sides of the △ EAB are proportionals, it is manifest from E. 4. 6., and E. 11. 5., that the sides of the △ FGH will, also, be proportionals.

PROP. LXXV.

85. THEOREM. *The semi-diameter of a given circle having been divided in extreme and mean ratio, the greater segment shall be equal to the side of an equilateral and equiangular decagon inscribed in the circle.*

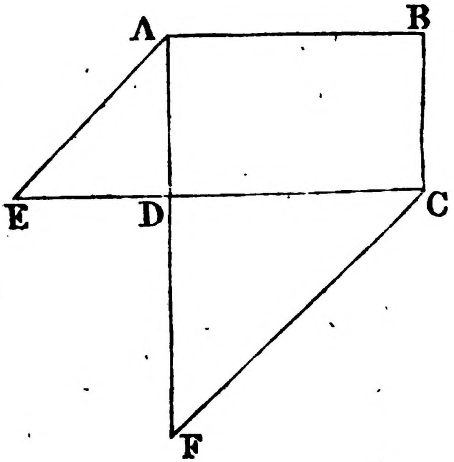
For (S. 14. 4. *cor.* 1.) if the semi-diameter of the circle be so divided, as that the rectangle, contained by the whole and the lesser part, may be equal to the square of the greater part, that greater segment will be equal to the side of an equilateral and equiangular decagon to be inscribed in the given circle; and (E. 17. 6. E. 3.

def. 6) when the semi-diameter has been so divided, it is cut in extreme and mean ratio.

PROP. LXXVI.

86. THEOREM. *Any rectangle is the half of the rectangle contained by the diameters of the squares of its two sides.*

Let ABCD be any given rectangle; produce



AD to F, and make DF = DC; produce, also, CD to E, and make DE = DA; join A, E and C, F; ∴ AE and CF are the diameters of the squares of AD and DC: Then is the rectangle ABCD equal to the half of $\overline{AE} \times \overline{CF}$.

For (*constr*. E. 5. 1. and E. 32. 1.) the two △ ADE, CDF, are equiangular;

∴ (E. 4. 6.) CF : AE :: CD : DE or DA;

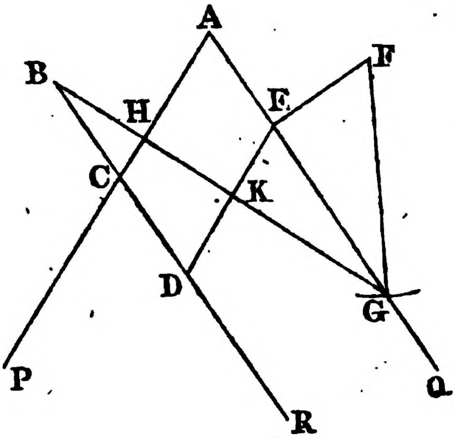
∴ the rectangle contained by CF and AE is (E. 1. def. 6.) similar to the rectangle contained by CD and DA; and since (*hyp.* and E. 10. def. 1.) the ∠ FDC is a right ∠, ∴ the rectangle $\overline{CF}\times\overline{AE}$, which is on CF, is equal (E. 31. 6.) to the two similar rectangles $\overline{CD}\times\overline{DA}$ and $\overline{FD}\times\overline{DE}$, which are on the equal sides CD and DF; that is, the rectangle $\overline{CF}\times\overline{AE}$ is double of the rectangle $\overline{CD}\times\overline{DA}$; or this latter rectangle is equal to the half of the former.

Prop. LXXVII.

87. Problem. *Through a given point, to draw a straight line, cutting two given straight lines, which meet one another, so that the triangle contained by the segment of that line and the segments which it cuts off from the given lines, shall be equal to a given rectilineal figure.*

Let AP and AQ be two given straight lines, which meet in A, and, first, let B be a given point without the ∠ PAQ: It is required to draw through B a straight line cutting AP and AQ, so that the △, contained by the segments of the three lines, shall be equal to a given square.

Through B draw (E. 31. 1.) BR parallel to AQ, and let it meet AP in C; to AC apply (E. 45. 1. *cor.*) the ▱ ACDE, having the ∠ ACD for one



of its ▱, and equal to the given figure; from E draw (E. 11. 1.) EF ⊥ to AQ, and make *it* equal to BC; from the centre F, at a distance equal to BD, describe a circle cutting AQ in G, and join F, G; ∴ FG = BD; lastly draw BG, cutting AP in H: Then is BG the line which was to be drawn.

For, let BG cut ED in K; the △ BCH, BDK, EKG, are (*constr.* E. 29. 1. and E. 15. 1.) equiangular, and ∴ (E. 4. 6.) similar to one another; and (*constr.*) BC = EF, BD = FG, and the ∠ FEG is a right ∠; it is manifest, ∴, that the three homologous sides BC, BD, and EG, of the three similar △ BCH, BDK, EKG, are the sides of a right-angled △; ∴ (E. 31. 6.) the △ BDK = △ BCH + △ EKG; take away the △ BCH, and there remains the trapezium HCDK = △ EKG; add to both these the trapezium AHKE, and it is evident that the △ AHG is equal to the ▭ ACDE, or (*constr.*) to the given square.

And, in a similar manner may the problem be solved, when the given point is within the given rectilineal ∠ PAQ.

88. Cor. Hence, and from S. 21. 6., a straight line may be drawn through a given point, which shall cut off from a given △ any required part of it.[*]

Prop. LXXVIII.

89. Problem. *Through a given point to draw a straight line, so as to cut off from two straight lines, that meet one another, two segments, toward their point of concourse, which shall contain a rectangle equal to a given square.*

Let the two given straight lines AP and AQ meet in A; and let B be a given point either within or without the ∠ PAQ: It is required to draw through B a straight line, so as to cut off from AP and AQ two segments, towards A,

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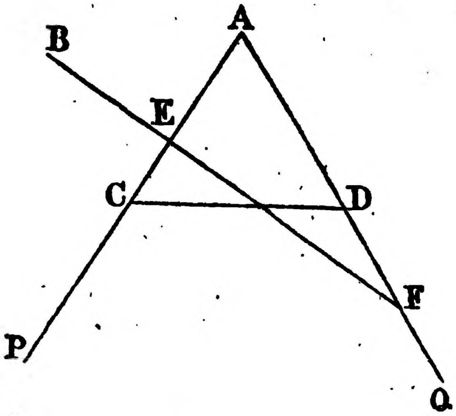
[*] Hence, and by the help of Trigonometry, any given rectilineal figure may be divided into two parts, which are to each other in any given ratio, by a straight line drawn from a given point, situated without the given figure.



which shall contain a rectangle equal to a given square.

From AP and AQ cut off AC and AD each of them equal to a side of the given square, and join C, D; through B draw (S. 77. 6.) $\overline{BEF}$ cutting off the △ AEF, equal to the △ ACD: Then, since the two △s EAF, CAD, have the same vertical ∠, ∴ (S. 5. 6. *cor.*)

$\triangle EAF : \triangle CAD :: \overline{EA} \times \overline{AF} : \overline{CA} \times \overline{AD}$ or $\overline{AC}^2$;

But (*constr.*) the △ EAF = △ ACD;

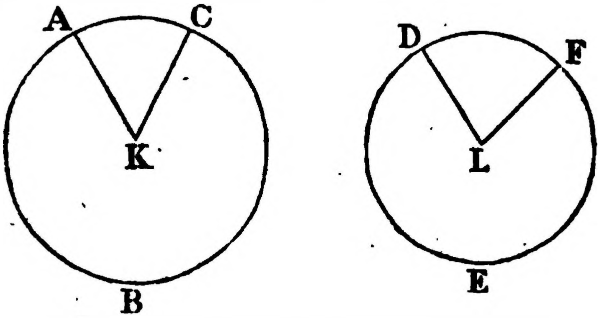
∴ $\overline{EA} \times \overline{AF} = \overline{AC}^2$; *i. e.* (*constr.*) the rectangle $\overline{EA} \times \overline{AF}$ is equal to the given square.

PROP. LXXIX.

90. THEOREM. *In different circles the semi-diameters which bound equal sectors contain angles*

reciprocally proportional to their circles; and conversely.

In the circles ABC, DEF, of which ABC is the



greater, first, let AKC, DLF, be two equal sectors: Then shall the ∠ AKC be to the ∠ DLF, as the circle DEF is to the circle ABC.

For (E. 33. 6.)

∠ AKC : four right ∠∠ :: sector AKC : circle ABC;
and,
four right ∠∠ : ∠ DLF :: circle DEF : sector DLF:
But (*hyp.*) sector AKC = sector DLE;
∴ (E. 23. 5.)
∠ AKC : ∠ DLE :: circle DEF : circle ABC.

Secondly, let the ∠ AKC be to the ∠ DLE, as the circle DEF is to the circle ABC: Then shall the sector AKC be equal to the sector DLE.

For it may be shewn as before that
four right ∠∠ : ∠ DLF :: circle DEF : sector DLF;
and (*hyp.* and S. 3. 5.)
∠ DLF : ∠ AKC :: circle ABC : circle DEF;
and
∠ AKC : four right ∠∠ :: sector AKC : circle ABC;

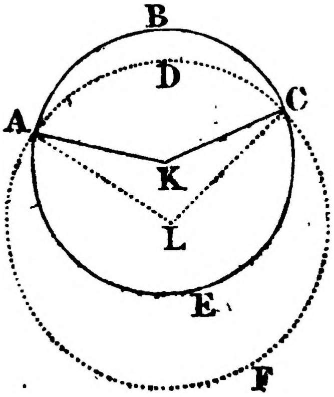
∴ (E. 23. 5.) four right ∠∠ : four right ∠∠ :: sector AKC : sector DLF;

∴ sector AKC = sector DLF.

Prop. LXXX.

91. **Problem.** *A given space being bounded by two arches of circles, subtending, at their respective centres, angles reciprocally proportional to the circles, to find a square that shall be equal to it.*

Let the space ABCDA be bounded by arches



ABC, ADC, of circles ABCE, ADCF, the centres of which are K and L; let the ∠ AKC be to the ∠ ALC, as the circle ADCF is to the circle ABCE: It is required to find a square that shall be equal to the space ABCDA.

Since (*hyp.* and S. 79. 6.) the sector ABCK is equal to the sector ADCL, from these equals take away the common part ADCK; and there remains the figure ABCDA equal to the rectilineal figure AKCL: Find, ∴, (E. 14. 2.) a square equal to the figure AKCL, and the problem will have been solved.

PROP. LXXXI.

92. PROBLEM. *To trisect a given circle, by dividing it into three equal sectors.*

Inscribe (E. 1. 1. E. 2. 4.) in the given circle an equilateral △: Then it is manifest, from E. 28. 3. and E. 33. 6., that the straight lines drawn from the centre of the circle to the three angular points of the inscribed △, will divide the circle into three equal sectors.

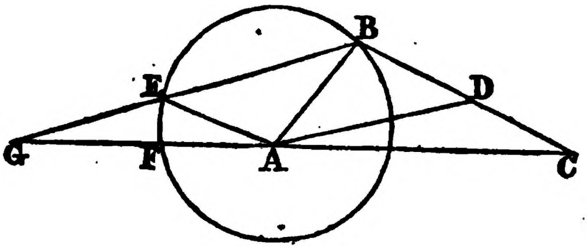
93. COR. In like manner, a circle may be divided into any required number of equal sectors, in all cases in which an equilateral figure, having that same number of sides, can be inscribed in the circle.

PROP. LXXXII.

94. THEOREM. *If, from the greater of two unequal*

sides of a given triangle, be cut off a part equal to the less, that segment shall have to the remaining segment, a ratio greater than the ratio which the angle adjacent to the remaining segment, has to the angle adjacent to the segment first cut off.

Let the side AB, of the triangle ABC, be less



than the side BC, and from BC let there be cut off BD = AB: Then (BD:DC) > (∠ ACB : ∠ ABC).

For draw $\overline{\text{AD}}$, and complete (E. 31. 1.) the ▭ ADBE; from the centre A, at the distance AB, describe the circle BEF, the circumference of which, since (E. 34. 1. and *constr.*) AE ≐ BD or AB, will pass through E; produce CA to meet the circumference BEF in F, and BE produced in G; so that (E. 29. 1.) the △s GEA, GBC, are equiangular.

Then, since the sector AEF is less than the △ AEG, and the sector AEB greater than the △ AEB,

$\therefore$ ($\triangle$ AEG : sector AEF) $>$ ($\triangle$ AEB : sector AEB);
$\therefore$ (S. 7. 5.)
($\triangle$ AEG : $\triangle$ AEB) $>$ (sector AEF : sector AEB):
But (E. 1. 6.) $\triangle$ AEG : $\triangle$ AEB :: GE : EB,
and (E. 4. 6. and E. 34. 1.) GE : EB :: BD : DC;
$\therefore$ (BD : DC) $>$ (sector AEF : sector AEB).
Also (E. 33. 6.)
sector AEF : sector AEB :: $\angle$ EAF : $\angle$ EAB;
and (E. 29. 1.)
the $\angle$ EAF $=$ $\angle$ ACB, and the $\angle$ EAB $=$ $\angle$ ABC;
$\therefore$ sector AEF : sector AEB :: $\angle$ ACB : $\angle$ ABC;
$\therefore$ (BD : DC) $>$ ($\angle$ ACB : $\angle$ ABC).

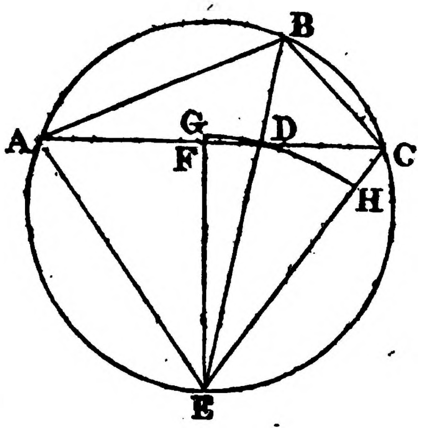
95. Cor. If any part BP be taken of BC, that is greater than AB, then, much more, is (BP : PC) $>$ $\angle$ ACB : $\angle$ ABC.)

Prop. LXXXIII.

96. Theorem. *The greater of any two unequal arches, of a given circle, has a greater ratio to the less arch, than the chord of the greater has to the chord of the less.*

Let $\widehat{AB}$ and $\widehat{AC}$ be any two unequal arches of the circle AECB, and let $\widehat{AB}$ be the greater: Then ($\widehat{AB}:\widehat{BC}$) $>$ ($\overline{AB}:\overline{BC}$).

For join A, C; bisect (E. 9. 1.) the $\angle$ ABC by BDE, and let BDE cut AC in D, and the circumference in E; join, also, E, C, and E, A; from E draw (E. 12. 1.) EF $\perp$ to AC.



And, since (*constr.* and E. 26. 3.) $\overset{\frown}{CE} = \overset{\frown}{AE}$, $\therefore$ (E. 29. 3.) $\overline{CE} = \overline{AE}$; and $\overline{EF}$ is common to the two right-angled $\triangle$ AFE, CFE; $\therefore$ (S. 73. 1.) $\overline{AF} = \overline{FC}$, and $\therefore$ $\overline{AD} > \overline{DC}$; also (E. 19. 1.) $\overline{EC} > \overline{ED}$, and $\overline{ED} > \overline{EF}$; if, $\therefore$, from the centre E, at the distance ED, a circle GDH be described, it will cut $\overline{EC}$, in H, between E and C, and $\overline{EF}$, produced, in F; so that the sector DEG $> \triangle$DEF, and the sector DEH $< \triangle$DEC;
$\therefore$ (sector DEG : sector DEH) $>$ ($\triangle$DEF : $\triangle$DEC):
$\therefore$ (E. 33. 6. and E. 1. 6.)

$$(\angle DEF : \angle DEC) > (\overline{DF} : \overline{DC});$$

$$\therefore \text{(S. 10. 5.)} \; (\angle FEC : \angle DEC) > (\overline{FC} : \overline{DC}).$$

In the same manner it may be shewn that

$$(\angle AEC : \angle DEC) > (\overline{AC} : \overline{DC});$$

$$\therefore \text{(S. 9. 5.)} \; (\angle AED : \angle DEC) > (\overline{AD} : \overline{DC});$$

$$\textit{i. e.} \text{ (E. 33. 6.)} \; (\overset{\frown}{AB} : \overset{\frown}{BC}) > (\overline{AD} : \overline{DC});$$

$$\therefore (\textit{constr.} \text{ and E. 3. 6.)} \; (\overset{\frown}{AB} : \overset{\frown}{BC}) > (\overline{AB} : \overline{BC}).$$

PROP. LXXXIV.

97. THEOREM. *The greater angle, at the base of a scalene triangle, has a greater ratio to the less angle, than the greater side has to the less side.*

Let ABC* be a scalene △; having the ∠ BCA greater than the ∠ BAC: Then $(\angle BCA : \angle BAC) > (\overline{AB} : \overline{BC})$.

About the △ ABC describe (E. 5. 4.) the circle AECB:

Then (E. 33. 6.) $\angle BCA : \angle BAC :: \overset{\frown}{AB} : \overset{\frown}{BC}$:

But (S. 83. 6.) $(\overset{\frown}{AB} : \overset{\frown}{BC}) > (\overline{AB} : \overline{BC})$;

$\therefore (\angle BCA : \angle BAC) > (\overline{AB} : \overline{BC})$.

PROP. LXXXV.

98. PROBLEM. *To divide a given circle into any required number of equal parts, by the circumferences of circles described within it, about its centre.*

Let ABC be a given circle: It is required to

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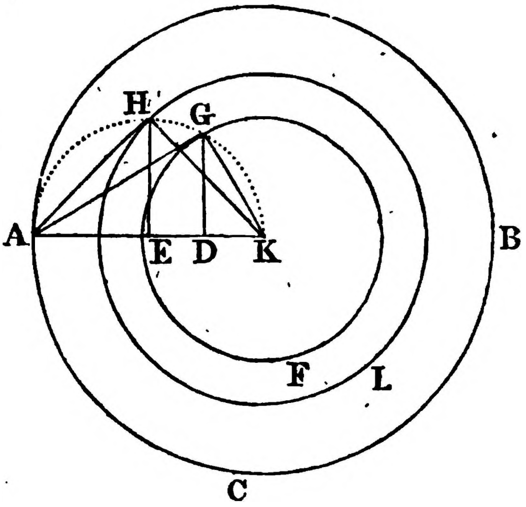
* See the figure in p. 402.



divide it into any number of equal parts, by the circumferences of circles described about the same centre with it.

Find (E. 1. 3.) the centre K of the circle ABC; take any semi-diameter KA; divide (S. 49. 1.) KA into as many equal parts, in the points D, E, &c. as those into which the given circle is to be divided; upon AK, as a diameter, describe the circle AHK; from D, E, &c. draw (E. 11. 1.) DG, EH, &c. ⊥ to AK, and meeting the circumference of AHK in G, H, &c.; join K, G and K, H, &c.; from the centre K, at the distances KG, KH, &c. describe the circles GF, HL, &c.: Then is the given circle divided in the required number of equal parts, by the circumferences of the circles so described.

For, join A, H, and A, G; and since (*constr.* and E. 31. 3.) the ∠∠ AHK, AGK, are right ∠∠, and HE, GD are ⊥ to AK,

∴ (E. 8. 6. and E. 20. 6. *cor.* 2.)

$$\overline{AK} : \overline{KD} :: \overline{AK}^2 : \overline{KG}^2 :$$

But (E. 2. 12. and E. 22. 6.)

$\overline{AK}^2 : \overline{KG}^2 ::$ circle ABC : circle GF :

∴ (E. 11. 5.) $\overline{AK} : \overline{KD} ::$ circle ABC : circle GF :

Likewise, $\overline{KE} : \overline{AK} ::$ circle HL : circle ABC;

∴ (E. 23. 5.) $\overline{KE} : \overline{KD} ::$ circle HL : circle GF;

∴ (E. 17. 5.)

$\overline{KD} : \overline{DE} ::$ circle HL—circle GF : circle GF;

But (*constr.*) KD = DE; ∴ the circle GF is equal to the space included between the circumferences of GF and HL: And, in the same manner, it may be shewn that this space is equal to the space included between the circumferences of HL, and of the circle next described according to the construction; and so on; ∴ the circle ABC is thus divided into the required number of equal parts, by the circumferences of circles that have the same centre with it.

Prop. LXXXVI.

99. Problem. *To find a circle, which shall be equal to the excess of the greater of two given circles above the less.*

Find (S. 75. 1.) a square which shall be equal to the excess of the square of the diameter of the greater circle above the square of the diameter of the less: It is manifest from E. 2. 12. and E. 17. 5., that the side of the square so found will be the diameter of the circle, which is equal to the excess of the greater of the given circles above the less.

Prop. LXXXVII.

100. Problem. *To find a circle to which a given circle shall have the same ratio, as that which one given straight line has to another.*

Find (E. 12. 6.) a fourth proportional (L) to the two given straight lines (A) and (B) and to the diameter (D) of the given circle; find, also, (E. 13. 6.) a mean proportional (M) between the diameter (D) of the given circle, and the fourth proportional (L) first found;

∴ (E. 20. 6. *cor.* 2.) $\overline{D}^2 : \overline{M}^2 :: D : L$;
and (*constr.*) $D : L :: A : B$;
∴ (E. 11. 5.) $\overline{D}^2 : \overline{M}^2 :: A : B$;

∴ (E. 2. 12.) the given circle has to a circle described on $\overline{M}$, as a diameter, the same ratio as that which A has to B.

PROP. LXXXVIII.

101. THEOREM. *If, in any given circle, two chords cut each other at right angles, the four circles described upon their segments, as diameters, shall, together, be equal to the given circle.*

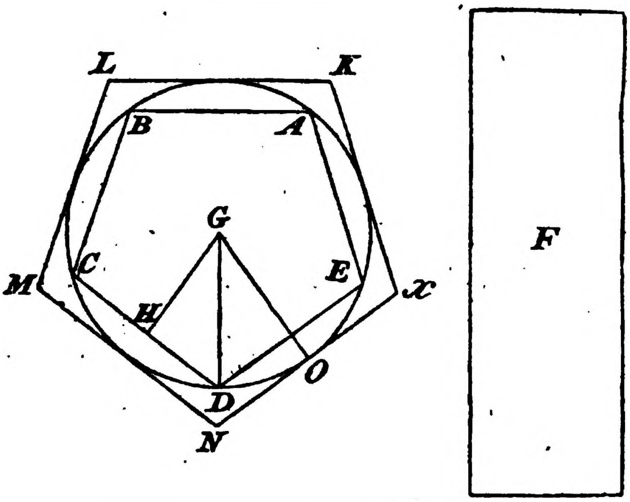
For (S. 50. 3.) the squares of the four segments are, together, equal to the square of the diameter: It is manifest, ∴, from E. 18. 5., and E. 2. 12., that the circles described on the four segments of the chords are, together, equal to the given circle.

PROP. LXXXIX.

102. THEOREM. *A circle is equal to the half of the rectangle contained by its semi-diameter and by a straight line which is equal to its circumference.*

Let ABCD be a circle, and let F be the half of the rectangle contained by its semi-diameter and by a straight line equal to its circumference: The circle ABCD is equal to the rectangle F.

For if it be not equal, it is either greater, or less,



than it. If it be possible, let F < the circle ABCD; ∴ (E. 2. 12.) a polygon ABCDE may be inscribed in the circle, which shall be greater than F.

Find (E. 1. 3.) the centre G, and from G draw (E. 12. 1.) $\overline{GH}$ ⊥ to any side CD, of ABCDE, and join G, D. Then it may be assumed that the circumference of the circle is greater than the perimeter of the inscribed figure ABCDE; and (E. 17. and 19. 1.) $\overline{GD} > \overline{GH}$; ∴ the rectangle contained by the circumference and the semidiameter of the circle is greater than that contained by GH, and the perimeter of ABCDE, which latter rectangle (E. 41. 1. and E. 1. 2.) is the double of the polygon ABCDE; ∴ F > ABCDE; and it is also less; which is absurd.

But, if it be possible, let F be greater than the circle. Then (E. 2. 12.) a polygon KLMNX may be described about the circle, which shall be less than F; join the centre G, and any of the points of contact O; and since it may be assumed that the perimeter of KLMNX is greater than the circumference of the circle, the rectangle contained by the perimeter of KLMNX and $\overline{GO}$, which rectangle is the double of KLMNX, is greater than the rectangle contained by the circumference of the circle and GO; ∴ the circumscribed polygon KLMNX > F; and it is also less; which is absurd. Therefore, the circle ABCD can neither be greater, nor less, than F; *i. e.* it is equal to F.

103. Cor. The circumferences of circles are to one another as their semi-diameters.

Prop. XC.

104. Theorem. *A circle is a mean proportional between any regular polygon, described about it, and a similar polygon, the perimeter of which is equal to the circumference of the circle.*

For if there be taken a straight line (P) equal to the perimeter of the regular polygon described about the circle, and another straight line (p) equal to the perimeter of the similar polygon, or (*hyp.*) equal to the circumference of the circle, then (E. 20. 6. and E. 22. 6.) the polygon, described about the circle, is to the similar polygon, as $\overline{P}^2$ is to $\overline{p}^2$: But (S. 2. 4. *cor.* 2.) the polygon, described about the circle, is the half of the rectangle contained by P and the circle's semi-diameter; and (S. 89. 6.) the circle is the half of the rectangle contained by *p*; and by the circle's semi-diameter; ∴ (E. 1. 6.) that polygon is to the circle, as P is to *p*; and it has been shewn to be to the similar polygon, as $\overline{P}^2$ is to $\overline{p}^2$; ∴ it has to the similar polygon a ratio, the duplicate of that which it has to the circle; ∴ the circle is a mean proportional between the two similar polygons.

THE END.

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